and 9 to give the same result.) If equation 2 is not satisfied by the approximate spectra, it seems reasonable to use equation 6 in preference to equation 9, since equation 6, obtained by integrating over equation 2, preserves the current. It is emphasized that if the flux spectrum is spatially dependent, the proper weight function for the parallel averaging procedure, equation 6, is the gradient of the flux, and not the flux itself. For multidimensional problems this will, in general, lead to an anisotropic diffusion coefficient.

For completeness, we show how one can arrive at another averaging scheme often quoted in the literature—that of averaging the reciprocal of the transport cross section in a homogeneous region over the Laplacian of the flux. Using equation 2 in equation 1, we find for the leakage term in a homogeneous region

$$\text{Leakage} = \frac{1}{3\Sigma_{\text{tr}}(E)} \nabla^2 \phi(\vec{r}, E). \tag{17}$$

Equating the integral over energy of equation 17 to the i^{th} group leakage term yields

$$(\text{Leakage})_i = \frac{1}{3}\left[\frac{1}{\Sigma_{\text{tr}}(E)}\right]_i^{\nabla^2} \nabla^2 \Phi^i(\vec{r}), \tag{18}$$

where we have defined

$$\left[\frac{1}{\Sigma_{\text{tr}}(E)}\right]_i^{\nabla^2} \equiv \frac{\int_i dE\left(\frac{1}{\Sigma_{\text{tr}}(E)}\right)\nabla^2\phi(\vec{r}, E)}{\int_i dE \nabla^2 \phi(\vec{r}, E)}. \tag{19}$$

Equation 19 indeed indicates that one should average the reciprocal of the transport cross section over the Laplacian of the flux. We note that this type of average leads to a simpler result than equation 6 in that the group-averaged diffusion coefficient is isotropic. This simplicity is obtained at the expense of accuracy—i.e., equation 19 was derived by integrating over the net leakage term, thus preserving the divergence of the current, whereas equation 6 was derived by integrating over Fick's law, equation 2, thus preserving the current itself. Accordingly, *a priori* one should expect that equation 6, which preserves more detailed quantities, will yield better over-all results than equation 19. If one is only interested in computing the group fluxes, both averaging procedures should be equally accurate since it is only the divergence of the current that enters into the calculation. If, however, one wishes to compute the group currents, then the use of equation 6 should yield more accurate results. It is interesting to note that the group-averaged diffusion coefficient defined by equation 19 is properly a multiplier of the Laplacian in equation 18, even though, in general, equation 19 will lead to a spatially dependent quantity. If equation 6 or equation 9 is used, the diffusion coefficient is acted upon by the divergence operator in computing the net leakage.

It is hoped that the foregoing remarks may be of some help in comparing the bases for the various averaging procedures suggested in the literature.

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Generalizations of Fick's Law

Using the one-velocity Boltzmann equation for slab geometry and a homogeneous isotropic medium, Adler[1] derived a relationship between the net current $J(z)$ and the neutron density $N(z)$,

$$J(z) = -\frac{v}{\Sigma_{tr}}\frac{d}{dz}\left[\overline{\mu^2}(z)N(z)\right] \tag{1}$$

$$\overline{\mu^2}(z) = \frac{\int \mu^2 n(z,\mu)d\mu}{\int n(z,\mu)\,d\mu} \tag{2}$$

$$\Sigma_{tr} = \Sigma - \overline{\mu}_0\,\Sigma_s \tag{3}$$

where $\overline{\mu^2}(z)$ is the mean square cosine of the angular distribution and $\overline{\mu}_0$ is the mean cosine of the scattering angle. Equation (1) is a one-velocity generalization of Fick's law.

The one-velocity restriction on Equation (1) can be removed by considering the velocity-dependent Boltzmann equation

$$\frac{\partial n}{\partial t} = -v_z\frac{\partial n}{\partial z} - kn + \iiint n(z,v',\underset{\sim}{\omega}',t)\,v'\,\Sigma_s(\underset{\sim}{v}'\rightarrow\underset{\sim}{v})\,dv'd\omega' \tag{4}$$

where $n(z, v, \underset{\sim}{\omega}, t)dVdvd\omega$ is the number of neutrons in dV at z whose speeds are in dv at v and whose directions of motion lie in the solid angle $d\omega$ at $\underset{\sim}{\omega}$ at time t. The term $k = v\Sigma$ is the collision rate per neutron. If we assume that the cross section for velocity change $\Sigma_s(\underset{\sim}{v}'\rightarrow\underset{\sim}{v})$ depends only on the initial speed v', the final speed v, and $\underset{\sim}{\omega}'\cdot\underset{\sim}{\omega}$, then

[1]F. ADLER, in *Reactor Handbook*, (H. SOODAK, ed.), AECD-3645, Vol. I, p. 385. United States Atomic Energy Commission (1955); *Reactor Handbook* (H. SOODAK, ed.), Vol. III, Part A, p. 140. Interscience Publishers, New York (1962).

$$\iiint \Sigma_s(\underset{\sim}{v}'\rightarrow\underset{\sim}{v})\, v\, dv\, d\omega = v'\alpha(v')\,\Sigma_s(v') \tag{5}$$

where $v'\alpha(v')$ is the mean component of velocity of the scattered neutron along the original direction of motion. If the speed is not changed by the collision, as in the one-velocity Boltzmann equation, then $\alpha = \overline{\mu}_0$.

A derivation of a generalization of Equation (1) from Equation (4) is obtained by noting that the pressure exerted by neutrons at z is given by

$$P(z) = m\iiint v_z^2 n dv d\omega \tag{6}$$

where m is the mass of a neutron. Thus, multiplying Equation (4) by $v_z = \mu v$, integrating over all speeds and angles, and using Equation (5) we obtain a relation between the neutron pressure and net current

$$\frac{\partial J}{\partial t} = -\frac{1}{m}\frac{\partial P}{\partial z} - \overline{k}_{tr}\, J \tag{7}$$

where $J(z) = \iiint v_z n dv d\omega$

$$\overline{k}_{tr}(z) = \frac{\iiint v_z n[k - \alpha\, k_s]\, dv\, d\omega}{\iiint v_z n\, dv\, d\omega} .$$

In the steady state, Equation (7) reduces to

$$J(z) = -\frac{1}{m\,\overline{k}_{tr}(z)}\;\frac{\partial P}{\partial z} \tag{8}$$

which resembles the equation of Knudsen flow[2]. Combining Equation (8) with the mean square z-component of velocity at position z defined by

$$\overline{v_z^2}(z) = \frac{P(z)}{m\,N(z)} = \frac{\iiint v_z^2 n\, dv\, d\omega}{\iiint n\, dv\, d\omega} \tag{9}$$

where $N(z) = \iiint n dv d\omega$, (10)

we obtain a generalization of Equation (1) which is not limited by the one-velocity restriction.

$$J(z) = -\frac{1}{\overline{k}_{tr}(z)}\;\frac{\partial}{\partial z}\,[\overline{v_z^2}(z)\; N(z)] . \tag{11}$$

If the neutron distribution is isotropic then $\overline{v_z^2} = \overline{v_x^2} = \overline{v_y^2}$, and since $\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = \overline{v^2}$, we have $\overline{v_z^2} = \overline{v^2}/3$. In the one-velocity case the transport collision rate $\overline{k}_{tr}(z)$ reduces to

$$\overline{k}_{tr} = v\,\Sigma_{tr} = v[\Sigma - \mu_0\,\Sigma_s] . \tag{12}$$

Thus, Equation (11) reduces to Equation (1), which in turn reduces to Fick's law

$$J = -\frac{v}{3\,\Sigma_{tr}}\;\frac{\partial N}{\partial z} \tag{13}$$

[2]E. H. KENNARD, *Kinetic Theory of Gases*, p. 304. McGraw-Hill, New York, (1938).

if the angular distribution is approximately isotropic.

Equations (1) and (11) can be used in cases that are beyond the validity of Fick's law. For example let us consider a mono-energetic unidirectional plane neutron source at $z = 0$ supplying S neutron per cm^2 per second in the z-direction in a purely absorbing medium. Since Σ_a is not considerably smaller than Σ_s, Fick's law is not applicable, but we can still use Equation (1) or (11). Here $\overline{v_z^2} = v^2$ so that they reduce to

$$J = -\frac{v}{\Sigma_a}\;\frac{dN}{dz} . \tag{14}$$

Combining Equation (14) with the neutron-conservation equation

$$\frac{\partial J}{\partial z} + \Sigma_a v\; N = 0 , \tag{15}$$

we obtain

$$N(z) = \frac{S}{v}\,\exp\,[-\Sigma_a z] . \tag{16}$$

The derivation of this well known result demonstrates the extended validity of these generalized forms of Fick's law.

Next we consider the neutron density generated by a distant plane neutron source producing S neutrons per cm^2 per second in a non-absorbing homogeneous medium filling the half-space $z \geq 0$. Let us treat the one-velocity case. If we apply Fick's law (13) to this problem we have

$$N(z) = \frac{3\,S\,\Sigma_{tr}}{v}\;[z + a] \tag{17}$$

where a is a constant of integration. This result states that the density decreases linearly as the boundary is approached. Now applying Equation (1) to this problem we obtain

$$N(z) = \frac{1}{\overline{\mu^2}(z)}\;\frac{S\,\Sigma_{tr}}{v}\;[z + a] . \tag{18}$$

When z is not near the boundary, the angular distribution of neutrons is approximately isotropic so that $\overline{\mu^2} = \frac{1}{3}$, and Equations (17) and (18) agree. When we approach the boundary the angular distribution becomes peaked in the direction towards the boundary and consequently $\overline{\mu^2}(z)$ increases. The result (Equation (18)) states that if $\overline{\mu^2}(z)$ increases, the density $N(z)$ decreases more sharply than Equation (17) near the surface of the medium. This simple qualitative argument agrees with exact transport-theory calculations.

In the above example the well known fact that $\overline{\mu^2}$ is not 1/3 near the boundary implied a sharp density decrease in this region. The argument may be reversed so that a knowledge of the neutron density $N(z)$ yields an immediate calculation of $\overline{\mu^2}(z)$. Combining the expression for the asymptotic neutron density,

$$N_{as}(z) = \frac{3\,S\,\Sigma_{tr}}{v}\,[z + a] \tag{19}$$

TABLE I

Mean Square Cosine of the Angular Distribution

$\Sigma_s z$	$\overline{\mu^2}(z) = N_{as}(z)/3\,N(z)$
0.0	.410
0.05	.384
0.1	.371
0.2	.357
0.5	.342
1.0	.336
2.0	.334
3.0	.333

with Equation (18), we obtain

$$\overline{\mu^2}(z) = \frac{1}{3}\,\frac{N_{as}(z)}{N(z)}\,. \tag{20}$$

Values of $\overline{\mu^2}(z)$ calculated from $N_{as}(z)$ and $N(z)$ given by Mark[3] are shown in Table I.

Thus, a generalization of Adler's form of Fick's law which uses the concept of neutron pressure has been derived, and several simple examples have been given to show that Equations (1) and (11) are generalizations of Fick's law which possess its simplicity and have a wider range of validity.

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[3] C. MARK, *Phys. Rev.*, **72**, 558 (1947).