On the Asymptotically Correct Approximation to the Transport Equation

A new truncation scheme for the spherical harmonics expansion has been proposed by Pomraning[1]. It yields the exact transport-theory asymptotic behavior in any homogeneous region. Pomraning treated the isotropic scattering case extensively and showed the extension to linearly anisotropic scattering. Comparison of his truncation scheme with the recent developments of the Case method for an arbitrary scattering law[2] brings out interesting similarities between exact and approximate methods in the general case. In this note the explicit expression for the ratio of the $N+1$'st to the $N-1$'st angular moment is given and the missing part of Pomraning's arguments is completed. Namely, it will be shown that the attenuation distances of the remaining solutions in the finite-order approximations are all shorter than the diffusion length as given by the asymptotic eigenvalue.

To avoid the ambiguities, a brief review of the regular solutions of the transport equation in one dimension is given first. The same notation as is standard in the Case method is used here. It is sufficient to demonstrate the relations in one dimension because the general solution in three dimensions, at least in principle, can be obtained by the superposition of elementary solutions along all possible directions. The transport equation is

$$[\mu(\partial/\partial x)+\Sigma]\,\phi(x,\mu)=\Sigma\int_{-1}^{1} d\mu' f(\mu\to\mu')\,\phi(x,\mu') \quad , \qquad (1)$$

where the scattering kernel is given by a finite series of Legendre polynomials

$$f(\mu_0)=\tfrac{1}{2}\sum_{m=0}^{M}(2m+1)f_m P_m(\mu_0); \qquad 0<f_0<1 \quad ;$$
$$|f_m|\le f_0 \quad ; \qquad (2)$$

thus, we assume no multiplication. The analysis of the multiplying case follows the same pattern and is not included here. In a homogeneous medium we are looking for solutions in the form

$$\phi(x,\mu)=\exp(-\Sigma x/\nu)\,\phi_\nu(\mu) \qquad (3)$$

with $\nu\notin(-1,1)$. The regular eigensolution is

$$\phi_\nu(\mu)=\frac{\nu}{2}\,\frac{1}{\nu\mp\mu}\,g(\nu,\mu) \quad , \qquad (4)$$

where

$$g(\nu,\mu)=\sum_{m=0}^{M}(2m+1)f_m g_m(\nu)P_m(\mu) \quad . \qquad (5)$$

The functions $g_m(\nu)$ are defined by the recursion formula

$$(n+1)g_{n+1}(\nu)+n g_{n-1}(\nu)-(2n+1)(1-f_n)\nu g_n(\nu)=0\,;$$
$$n\le M \quad ;$$
$$g_0=1 \quad . \qquad (6)$$

We note that all angular moments of the regular eigensolution for $n\geqslant M$ are proportional to $g(\nu,\nu)$ as it is deduced from Eqs. (4) and (5), namely[3],

$$g_n(\nu)=\nu\mathcal{Q}_n(\nu)g(\nu,\nu) \quad ; \qquad n\ge M \quad , \qquad (7)$$

where $\mathcal{Q}_n(\nu)$ are Legendre's functions of the second kind defined on the complex plane with the cut (-1,1).

The asymptotic (i.e., the discrete) eigenvalue $\pm\nu_0$ is determined as the zero of the dispersion function $\Lambda(z)$

$$\Lambda(z)=(M+1)\,[\mathcal{Q}_M(z)g_{M+1}(z)-\mathcal{Q}_{M+1}(z)g_M(z)] \qquad (8)$$

as derived from Eq. (7) for $n=M$ and $n=M+1$. The more familiar form of the dispersion function is obtained from Eq. (8) using recursion formulae for $\mathcal{Q}_n(z)$ and $g_n(z)$

$$\Lambda(z)=1-z\sum_{m=0}^{M}(2m+1)f_m g_m(z)\mathcal{Q}_m(z) \quad . \qquad (9)$$

Turning now to the spherical-harmonics expansion method and the new truncations scheme of Pomraning's in particular, we observe, first, that N, the order of the finite approximation, must be larger than or equal to M to be physically and computationally consistent. Second, the angular moments of the spherical-harmonics expansion up to $n=M$ obey the same recursion formula, Eq. (6), as functions g_n and for $n>M$ obey a simplified recursion formula, obtained from Eq. (6) by setting f_n equal to zero. To obtain the asymptotically correct approximation, it is only necessary to require that the angular moments do not violate Eq. (7) for $n>M$. This has been done by Pomraning by expressing the $N+1$'st angular moment by the $N-1$'st in the closing equation for $n=N$. From Eq. (7) one derives the general expression for his ratio α_N, which is

$$\alpha_N=\mathcal{Q}_{N+1}(\nu_0)/\mathcal{Q}_{N-1}(\nu_0) \qquad (10)$$

where we write now ν_0 for the exact value of the asymptotic eigenvalue to distinguish it from other eigenvalues, obtained in the finite-order approximations. Using the recursion formula for $\mathcal{Q}_N(\nu_0)$, the closing equation of the finite set of angular moments becomes for any $N>M$

$$\nu_0\mathcal{Q}_N(\nu_0)\Psi_{N-1}(\nu)-\nu\mathcal{Q}_{N-1}(\nu_0)\Psi_N(\nu)=0 \quad . \qquad (11)$$

Equation (11) is fully equivalent to the requirement that the determinant of the system of linear equations for the angular moments $\Psi_n(\nu)$ is equal to zero. For $\nu=\nu_0$ one can reduce Eq. (11), using recursion formulae, to the statement $\Lambda(\nu_0)=0$ in the same way as Eq. (9) has been derived from Eq. (8). This proves that, choosing α_N according to Eq. (10), one of the approximate solutions will decay with the distance away from the source in the same way as the exact asymptotic solution.

It remains to prove that ν_0 is the largest positive zero of Eq. (11), and, furthermore, that besides ν_0 there are exactly $\frac{1}{2}(N-1)$ pairs of zeros between $-\nu_0$ and ν_0 for odd N. It would be desirable to prove that the additional zeros are all confined to the interval (-1,1), yet this part of the general proof is still missing.

Following Davison[4], we introduce a new variable

$$y=1/\nu^2 \qquad (12)$$

and the functions

$$R_n(y)=(n+1)\Psi_{n+1}(\nu)/\nu\Psi_n(\nu) \quad . \qquad (13)$$

In terms of functions R_n, the recursion formula, Eq. (6), becomes

$$R_n(y)=(2n+1)(1-f_n)-n^2y/R_{n-1}(y) \quad ;$$
$$f_n=0 \text{ for } n>M \quad . \qquad (14)$$

[1]G. C. POMRANING, "An Asymptotically Correct Approximation to the Multidimensional Transport Equation," *Nucl. Sci. Eng.*, **22**, 3, 328 (1965).

[2]S. PAHOR and I. KUŠČER, "On the Non-Uniqueness of Solutions of Chandrasekhar's S-Equation in Radiative Transfer," *Astrophys. J.*, **143**, 888 (1966).

[3]I. S. GRADSHTEYN and I. M. RYZHIK, *Table of Integrals, Series and Products*, 4th ed., p. 821, Academic Press, New York (1965) (See integral **7.224.5**).

[4]B. DAVISON *Neutron Transport Theory*, Oxford University Press, Inc., New York (1957).

For the odd P_N approximation we obtain exactly $\frac{1}{2}(N+1)$ zeros $y^P_{N,m}$ by setting R_N equal to zero, as has been proved by Davison for the isotropic scattering. With our restrictions imposed upon the scattering law, Eq. (2), the proof remains unchanged in the case of anisotropic scattering. Thus, due to the ordering of the zeros for different orders of approximations,

$$y_0 < \ldots < y^P_{N+1,0} < y^P_{N,0} < y^P_{N-1,0} < \ldots \quad (15)$$

and the properties of $R_n(y)$ functions

$$R_n(0) > 0 \quad \text{and} \quad \frac{d}{dy} R_n(y) < 0 \quad , \quad (16)$$

we conclude that the finite approximation dispersion function, Eq. (11), expressed in terms of R_{N-1} function

$$R_{N-1}(y) = y\,T(\nu_0) \quad (17)$$

has no zeros between 0 and $y_0 = 1/\nu_0^2$, because the value of $T(\nu_0)$ given by

$$T(\nu_0) = N\nu_0 \mathcal{Q}_N(\nu_0)/\mathcal{Q}_{N-1}(\nu_0) \quad (18)$$

is positive, since it is an even function of ν_0. Thus, y_0 is the smallest zero of Eq. (17) and the first part is proved.

Observing that even-order functions R_{2n} have positive finite limits as y goes toward infinity, that they have negative derivatives, and that they have poles at zeros of the corresponding R_{2n-1} functions, we conclude, by repeating the arguments of Davison, that Eq. (17) has exactly $\frac{1}{2}(N+1)$ distinct zeros for N odd, including y_0. This concludes the proof. As an example, Fig. 1 illustrates the relationship among zeros for P_5 and A_5 approximations.

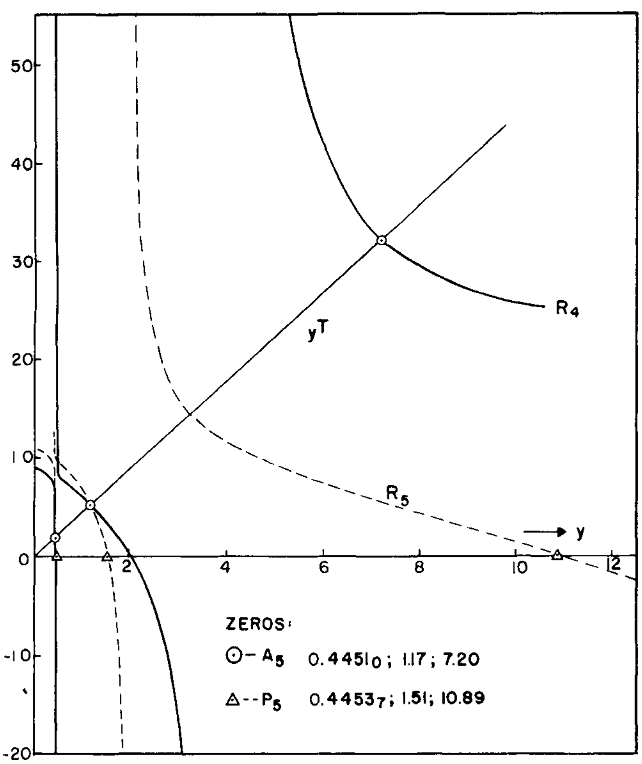

Fig. 1. Relative positions of zeros in P_5 and A_5 approximations for the scattering law with $f_0 = 0.75$; $f_1 = 0.25$; $f_2 = 0.10$; $f_3 = 0.05$; and $f_4 = 0.01$. If the interval around $y = 0.445$ is expanded, one obtains the picture of the first zeros that looks very similar to the situation around $y = 1.5$, where the second zeros are situated.

M. Čopič

Department of Nuclear Engineering
Kansas State University
Manhattan, Kansas

May 31, 1966

Comments on an Article by J. Nilsson and R. Sandlin

In a recent paper, Nilsson and Sandlin[1] have discussed use of a source separation technique for investigating the attenuation of neutrons in annular ducts. In methods of this type, one attempts to separate the neutron flux into its several components by use of thin cadmium sheets. Of course, it is only thermal (subcadmium) neutrons that can be successfully separated in this manner. However, such techniques are potentially extremely valuable for obtaining a better understanding of the behavior of radiation in shields containing ducts. It is unfortunate that the work reported by Nilsson and Sandlin does not take full advantage of the information available in a source separation experiment.

The first problem encountered in this paper is concerned with the experimental techniques used. The authors have claimed that the measurements "account only roughly for the components as defined. . . ." However, the quantities which were measured do not even "roughly" correspond to the definitions of the various flux components. For instance, the streaming component (S) is supposed to include only those neutrons which enter the duct mouth in the thermal energy range and travel to the detector without striking the duct walls. However, the quantity that was measured includes all neutrons which enter the duct mouth in the thermal energy range. The true streaming component could have been measured by lining both walls of the annular duct with cadmium.

In addition, the albedo component (A) is supposed to include neutrons of all energies which enter the duct mouth and strike the walls before returning to the duct as thermal neutrons. The experimental arrangements used in the measurement of the albedo component contained cadmium covers at the duct mouth. Therefore, thermal neutrons were prevented from entering the duct mouth during this part of the experiment.

The second major difficulty encountered is the large uncertainty associated with some of the data. Use of data not known to better than a factor of 10 makes meaningful comparison with theory nearly impossible. For the source separation technique to be of maximum value, it is necessary to measure each flux component with reasonable accuracy. This can be accomplished by maintaining sufficiently high count rates during the measurement of basic quantities so that even the derived flux components have a small associated standard deviation.

The third problem encountered in this paper is the multiplying constants which appear with the theoretical expressions for each flux component. The constants that were found to give best agreement can be expected to apply

[1]J. NILSSON and R. SANDLIN, "Measured and Predicted Thermal- and Fast-Neutron Fluxes in Air-Filled Annular Ducts," *Nucl. Sci. Eng.*, 23, 3, 224 (1965).