which is the usual result for constant heat capacity ($\gamma = 0$), and to

$$T_\infty = \tfrac{3}{2}\delta k_p/\alpha \quad \text{for} \quad \sigma = 0 \tag{14}$$

so that a 25% reduction in final fuel element temperature is possible for this case. The total energy release may be computed from (12) by noting that

$$E_\infty = C_0 T_\infty + (\gamma T_\infty^2/2). \tag{15}$$

The peak power occurs at $\theta = 1$, as is clear from (9), so that (11) yields

$$P_{\max} - P_0 = \frac{1 + 3\sigma}{6\sigma} \frac{C_0(\delta k_p)^2}{\alpha l}. \tag{16}$$

It is of interest to discuss the physical interpretation of the various limiting cases here also but, for brevity, we shall compare (16) with the constant heat capacity result

$$P_{\max}^{(F)} - P_0 = \frac{\bar{C}(\delta k_p)^2}{2\alpha l} = \frac{C_0(\delta k_p)^2}{2\alpha l} \frac{1 + \alpha}{\sigma} \tag{17}$$

where we have chosen an "average" heat capacity over the course of the pulse

$$\bar{C} = C_0 + \gamma\left(\frac{\delta k_p}{\alpha}\right) = C_0\left(1 + \frac{1}{\sigma}\right). \tag{18}$$

From (16) and (17)

$$\frac{P_{\max} - P_0}{P_{\max}^{(F)} - P_0} = \frac{1 + 3\sigma}{3(1 + \sigma)} \tag{19}$$

so that (neglecting P_0)

$$P_{\max}/P_{\max}^{(F)} = 1 \quad \text{for} \quad \sigma = \infty \tag{20}$$

as expected and

$$P_{\max}/P_{\max}^{(F)} = \tfrac{1}{3} \quad \text{for} \quad \sigma = 0, \tag{21}$$

i.e., a drop of 67% is possible with respect to the value obtained by using an average heat capacity. Needless to say, values for specific cases should be computed using the full formulas (12) and (16), and due regard must be paid to the validity of the linear approximation (1).

We may finally remark that substitution of (11) into (10) and a decomposition into partial fractions enables one to carry out the final integration which gives the temperature as a function of time; since the dependence is implicit, i.e., $f(T) = t$, where f is a transcendental function, we shall forego a detailed discussion (*3*). However, it may be noted that a few percent broadening of the pulse width, as compared to the constant heat capacity case, is a general characteristic of the solution.

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Green's Function for a Bare Slab with Anisotropic Scattering*

Under the assumption that the scattering function can be expanded into a finite series of legendre polynomials, the one-velocity Boltzmann equation in the case of plane symmetry has the form

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \psi(x, \mu) = \frac{c}{2} \sum_{k=0}^{N} b_k P_k(\mu) \int_{-1}^{1} P_k(\mu') \psi(x, \mu') \, d\mu' \tag{1}$$

where x is in terms of optical thickness (*1*), c is the mean number of secondarys which emanate from a neutron-nucleus interaction, and the b_k are the coefficients of the Legendre polynomial expansion. The general solution to this equation has been found by A. Jacobs (*2*) and J. Mika (*3*) in the form

$$\begin{aligned} \psi(x, \mu) &= \sum_{j=1}^{M} a_{+j} \phi(+L_j, \mu) e^{-x/L_j} \\ &+ \sum_{j=1}^{M} a_{-j} \phi(-L_j, \mu) e^{x/L_j} + \int_{-1}^{1} A(\nu) \phi(\nu, \mu) e^{-x/\nu} \, d\nu \end{aligned} \tag{2}$$

where the ϕ's are known functions and the coefficients $a_{\pm j}$ and $A(\nu)$ are determined from boundary conditions. The eigenfunctions have useful orthogonality properties and have been shown complete in the space of prescribed boundary variations, $\psi(\mu)$, which satisfy Eq. (1).

The Green's function problem for a bare slab of thickness T with a source plane at x_0 emitting neutrons at $\mu = \mu_0$ can be defined as follows

$$\psi^+(T, \mu) = 0 \qquad \mu < 0 \tag{3}$$

$$\psi^-(0, \mu) = 0 \qquad \mu > 0 \tag{4}$$

$$\mu[\psi^+(x_0, \mu) - \psi^-(x_0, \mu)] = \delta(\mu - \mu_0) \tag{5}$$

The quantities ψ^+ and ψ^- refer to the neutron distributions to the right and left of the source plane respectively.

Due to the nature of the boundary conditions, in the calculations that follow there will occur coefficients which are zero over half their respective ranges in ν. It will therefore be convenient to decompose the continuous coefficients in the eigenfunction expansion (2), into two half-range coefficients (*4*). Putting the discrete summations in a more compact notation, Eq. (2) may be written as

$$\begin{aligned} \psi^\pm(x, \mu) &= \sum_{\pm L_j}^{M} a_{\pm j}^{\pm} \phi(\pm L_j, \mu) e^{\mp x/L_j} \\ &+ \int_{-1}^{1} [A^\pm(\nu) h(\nu) + B^\pm(-\nu) h(-\nu) e^{b^\pm/\nu}] \phi(\nu, \mu) e^{-x/\nu} \, d\nu \end{aligned} \tag{6}$$

where $h(\nu)$ is the Heaviside step function and where

$$b^+ = T; \qquad b^- = x_0$$

Upon application of the orthogonality properties of the eigenfunctions Eq. (5) yields

$$A^+(\nu) - A^-(\nu) = \phi(\nu, \mu_0) e^{x_0/\nu}/M(\nu) \quad \nu > 0 \tag{7a}$$

$$B^+(\nu) e^{(x_0 - T)/\nu} - B^-(\nu) = \phi(-\nu, \mu_0)/M(-\nu) \qquad \nu > 0 \tag{7b}$$

$$a_{\pm j}^{+} - a_{\pm j}^{-} = \frac{\phi(\pm L_j, \mu_0)}{M_{\pm j}} e^{\pm x_0/L_j} \tag{7c}$$

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The normalizations $M_{j\pm}$, $M(\nu)$ and all further quantities not explicitly defined in this note are to be taken as defined by Mika (3).

Using the property $\phi(\nu, \mu) = \phi(-\nu, -\mu)$ and applying Eq. (3) yields the singular integral equation

$$\int_0^1 B^+(\nu)\phi(\nu, \mu)\, d\nu = \psi'(\mu) \qquad \mu > 0 \quad (8)$$

where the dominant part is given by

$$\psi'(\mu) \equiv -\sum_{\pm L_j}^{M} a_{\pm j}^{+}\phi(\mp L_j, \mu)e^{\mp T/L_j} - \int_0^1 A^+(\nu)\phi(-\nu, \mu)e^{-T/\nu}\, d\nu \quad (9)$$

A solution to this equation has been obtained in the form (2, 3)

$$B^+(\mu) = \mathbf{\Gamma}\psi''(\mu) \quad (10)$$

where the operator $\mathbf{\Gamma}$ is defined by the relation

$$\mathbf{\Gamma}\psi''(\mu) \equiv \frac{P\Omega(\mu)}{S(\mu)}\psi''(\mu) - \frac{P\Omega(\mu) - (i\pi c\mu/2)N(\mu)}{S(\mu)X^-(\mu)}\frac{c}{2}\mu N(\mu) \cdot \int_0^1 \frac{X^-(\nu)\psi''(\nu)}{[P\Omega(\nu) - (i\pi c\nu/2)N(\nu)]}\frac{d\nu}{(\nu - \mu)} \quad (11)$$

where $X(\nu)$, $P\Omega(\nu)$, $S(\mu)$, and $N(\nu)$ are all known functions and where

$$\psi''(\mu) \equiv \psi'(\mu) + \frac{c}{2}\sum_{n=0}^{N}\sum_{k=0}^{n}\sum_{m=0}^{k} b_n P_n(\mu)C_{nk}\mu^{k-m}g_m \quad (12)$$

The C_{nk} are some numbers and

$$g_m \equiv \int_0^1 B^+(\nu)\nu^m\, d\nu \quad (13)$$

Equation (10) is an inhomogeneous Fredholm integral equation with a degenerate kernel and may be readily reduced to a system of algebraic equations (6). It will be useful to rearrange the terms in the sum such that

$$\sum_{n=0}^{N}\sum_{k=0}^{n}\sum_{m=0}^{k} \to \sum_{m=0}^{N}\sum_{n=m}^{N}\sum_{k=m}^{n}$$

The solution of the Fredholm equation yields

$$g_m = D_m/D$$

where

$$D \equiv \begin{vmatrix} (1 - \Lambda_{00}) - \Lambda_{01} & \cdots & -\Lambda_{0N} \\ -\Lambda_{10}(1 - \Lambda_{11}) & \cdots & \vdots \\ \vdots & & \vdots \\ -\Lambda_{N0} & \cdots & (1 - \Lambda_{NN}) \end{vmatrix}$$

and

$$\Lambda_{nm} \equiv \int_0^1 \mu^n \Lambda_m(\mu)\, d\mu \equiv \int_0^1 \mu^n \mathbf{\Gamma}\frac{c}{2}\sum_{n=m}^{N}\sum_{k=m}^{n} b_n P_n(\mu)C_{nk}\mu^{k-m}\, d\mu$$

The determinant D_m is obtained by replacing the elements in the mth column Λ_{0m}, $\Lambda_{1m} \cdots \Lambda_{Nm}$ with the coefficients λ_0, λ_1, $\cdots \lambda_N$ where

$$\lambda_n \equiv \int_0^1 \mu^n \mathbf{\Gamma}\psi'(\mu)\, d\mu$$

Define the operator $\mathbf{P}$ with the following relation

$$\mathbf{P}\psi(\mu) \equiv -\mathbf{\Gamma}\psi(\mu) - \sum_{m=0}^{N}\frac{\Lambda_m(\mu)}{D}\sum_{j=0}^{N}(-1)^{j+m}M_{jm}\int_0^1 \nu^j\mathbf{\Gamma}\psi(\nu)\, d\nu$$

The quantity M_{jm} is the minor of the element λ_j obtained by expanding D_m in a Laplace development along the mth column. It follows that

$$B^+(\mu) = -\mathbf{P}\psi'(\mu)$$

Using Eq. (7a) this may be more explicitly written as

$$B^+(\mu) = \sum_{\pm L_j}^{M} a_{\pm j}^{+}e^{\mp T/L_j}\mathbf{P}\phi(\mp L_j, \mu) + \int_0^1 \frac{\phi(\nu, \mu_0)e^{(x_0 - T)/\nu}}{M(\nu)}\mathbf{P}\phi(-\nu, \mu\, d\nu) + \int_0^1 A^-(\nu)^{-T/\nu}\mathbf{P}\phi(-\nu, \mu)\, d\nu \quad (14)$$

Upon applying boundary condition (4) and following the same procedure used for $B^+(\mu)$ it follows that

$$A^-(\mu) = \sum_{\pm L_j}^{M} a_{\pm j}^{-}\mathbf{P}\phi(\pm(L_j, \mu) - \int_0^1 \frac{\phi(-\nu, \mu_0)e^{-x_0/\nu}}{M(-\nu)}\mathbf{P}\phi(-\nu, \mu)\, d\nu + \int_0^1 B^+(\nu)e^{-T/\nu}\mathbf{P}\phi(-\nu, \mu)\, d\nu \quad (15)$$

It is useful to note that the functions $\mathbf{P}\phi(-\nu, \mu)$ and $\mathbf{P}\phi(\pm L_j, \mu)$ are uniquely determined by the physical properties of the medium. The range of μ, (0, 1), is the only boundary condition inherent in these functions. They may consequently be used with arbitrary slab thicknesses and source positions. They may also be of use in other half-range problems.

Substitution of Eq. (15) into Eq. (14) will yield a Fredholm integral equation for the coefficient $B^+(\mu)$ which may be reduced by standard techniques. This solution in conjunction with Eqs. (15), (7a), and (7b) yields expressions for all the continuous coefficients in terms of the discrete ones. The problem is now reduced to finding sufficient conditions for the determination of the discrete coefficients. These conditions are provided by the operator $\mathbf{\Gamma}$ used to reduce Eq. (8) to a Fredholm integral equation. When the index of the integral equation is negative some added conditions are required if $\mathbf{\Gamma}$ is to have the proper behavior at infinity (5). The index of Eq. (8) is $-M$ making it necessary to impose the conditions (5)

$$\int_0^1 \frac{\nu^j X^-(\nu)\psi''(\nu)\, d\nu}{P\Omega(\nu) - (i\pi c\nu/2)N(\nu)} = 0 \qquad j = 0, 1, \cdots, (M - 1)$$

There will be an additional set of M equations obtained in the derivation of Eq. (15). Substitution of the solutions obtained for the continuous coefficients into these relations yields $2M$ algebraic equations for the $4M$ unknown a_j's. The additional $2M$ equations supplied by Eq. (7c) allow a complete determination of the discrete coefficients.

The Green's function problem for a bare slab has been in principle solved. The integrations involved are complicated and it is doubtful that they can be expressed in closed form. It is probable that the coefficients and the functions $\mathbf{P}\phi$ must be calculated numerically and then tabulated. Fortunately in many important cases the spherical harmonic expansion of the scattering distribution converges rapidly. For many problems of practical interest there are only one or two terms in the discrete set of functions, thereby reducing considerably the amount of algebra involved. In solving problems for a distributed source $S(\mu_0)$, caution must be exercised in choosing the order of integration where doubly singular integrals are involved. It has been shown that in some cases, integrations involving a Green's function may not be in the order that is usually expected (*2*). The solution to the albedo problem or the shielding problem for a slab with an incident neutron beam is obtained by placing the source plane at the origin.

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