Letters to the Editors

Tables of Secant Integrals of the First and Second Kinds

A class of integral functions which one frequently must evaluate in working shielding problems and other problems involving the attenuation of radiation through shields is the class known as the Secant Integrals.

This class of functions can be mathematically defined as

$$I_{n+1}(\theta, b) = b^n \int_0^\theta (\sec \theta')^n e^{-b \sec \theta'} \, d\theta', \quad n = 0, 1, 2, 3, \cdots \quad (1)$$

where

$b = \mu t$, single material
$b = \sum_i \mu_i t_i$, $i = 1, 2, \cdots$ for a number of layers
μ_i = mass absorption coefficient for the ith material
t_i = material thickness of the ith material.

$$I_1(\theta, b) \equiv F(\theta, b) = \int_0^\theta e^{-b \sec \theta'} \, d\theta. \quad (2)$$

Curves of these integrals have been published in Rockwell, "Shielding Design Manual," TID-7004 (1956).

$$I_2(\theta, b) \equiv G(\theta, b) = b \int_0^\theta \sec \theta' e^{-b \sec \theta'} \, d\theta'. \quad (3)$$

This class of integrals arises wherever a buildup function $B(b, \theta)$, which appears beneath the integral sign, is approximated by a polynominal expression. For example, if we have a function $\Gamma(\theta, b)$ such that

$$\Gamma(\theta, b) = \Gamma_0 \int_0^\theta B(b, \theta) e^{-b \sec \theta} \, d\theta \quad (4)$$

where $B(b, \theta)$ is approximated as

$$B(b, \theta) = 1 + \alpha b \sec \theta. \quad (5)$$

$$\alpha = \text{constant}.$$

From the definition of the $F(\theta, b)$ and $G(\theta, b)$ functions, Eqs. (2) and (3), Eq. (6) may be rewritten

$$\Gamma(\theta, b) = \Gamma_0[F(\theta, b) + \alpha G(\theta, b)]. \quad (6)$$

$$I_3(\theta, b) \equiv H(\theta, b) = b^2 \int_0^\theta \sec^2 \theta' e^{-b \sec \theta'} \, d\theta'. \quad (7)$$

These integrals arise when the expression for buildup is written

$$B(\theta, b) = 1 + \alpha b \sec \theta + \beta b^2 \sec^2 \theta; \alpha, \beta = \text{constant}. \quad (8)$$

Then the function $\Gamma(\theta, b)$ in Eq. (4) may be written

$$\Gamma(\theta, b) = \Gamma_0[F(\theta, b) + \alpha G(\theta, b) + \beta H(\theta, b)]. \quad (9)$$

The approximation given in Eq. (5) is sufficient for most

shielding calculations which are used for practical application. The secant integrals of the first and second kinds were computed by numerical integration of Eqs. (2) and (3) on an IBM-650 computer, and have been used extensively over the past several years in shielding calculations at Atomic Power Development Associates, Inc. They are available from that organization in tabular form.

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Effective Resonance Integral Dependence on the Moderator Slowing Down Properties

The effective resonance integral calculated according to the usual approximations NR (narrow resonance) or NRIA (narrow resonance infinite mass absorber) is independent of the moderator slowing down properties (e.g., *1*). As the calculation of the heterogeneous assembly is usually reduced to the calculation of the modified homogeneous mixture, the above statement holds in both cases. It also seems that beyond the experimental errors no influence of the moderator on the measured resonance integrals is found.

In this paper we wish to show by exact calculation of the resonance absorption in an infinite homogeneous mixture for resolved resonances of U^{238} how far the moderator slowing down properties and the interference scattering actually influence the effective resonance integral and its temperature coefficient. We will also compare the exact results with the usual NR and NRIA approximations.

The exact resonance absorption is calculated by numerical solution of the neutron slowing down equation for a mixture of elements:

$$F(u) = \sum_n \frac{1}{1 - \alpha_n} \int_{u-\epsilon_n}^u \frac{\Sigma_{sn}(u')}{\Sigma(u')} F(u') e^{u'-u} \, du'. \quad (1)$$

The symbols have the following meaning: $F(u) = \phi(u) \Sigma(u)$, the collision rate density; $\phi(u)$ is the flux of neutrons per unit lethargy at lethargy u; $\Sigma(u) = \sum_n [\Sigma_{sn}(u) + \Sigma_{an}(u)]$ is the total macroscopic cross section at lethargy u; $\Sigma_{sn}(u)$ is the macroscopic scattering cross section of the nth element including its potential scattering; $\Sigma_{an}(u)$ is the macroscopic absorption cross section of the nth element; $\alpha_n = [(A_n - 1)/(A_n + 1)]^2$, A_n is the atomic mass number of the nth element; $\epsilon_n = \ln 1/\alpha_n$. The suitable form for

numerical solution of (1) is obtained by differentiation of (1) with respect to lethargy. After some rearrangement one obtains

$$-\frac{dF(u)}{du} = \frac{\Sigma_a(u)}{\Sigma(u)} F(u) + \sum_n \frac{\alpha_n}{1 - \alpha_n}$$
$$\cdot \left[\frac{\Sigma_{sn}(u - \epsilon_n)}{\Sigma(u - \epsilon_n)} F(u - \epsilon_n) - \frac{\Sigma_s(u)}{\Sigma(u)} F(u) \right] \quad (2)$$

where $\Sigma_a(u) = \sum_n \Sigma_{an}(u)$. The corresponding equations of the NR and NRIA approximations are, respectively:

$$-\frac{dF(u)}{du} = \frac{1}{\xi} \frac{\Sigma_a(u)}{\Sigma(u)} F(u) \qquad (NR), \quad (3)$$

$$-\frac{dF(u)}{du} = \frac{1}{\xi^*} \frac{\Sigma_a(u)}{\Sigma^*(u)} F(u) \qquad (NRIA). \quad (4)$$

The symbols introduced are: $\bar{\xi} = \sum_n \xi_n \Sigma_{pn} / \sum_n \Sigma_{pn}$; ξ_n is the average logarithmic energy decrement of the nth element; Σ_{pn} is the macroscopic potential scattering cross section of the nth element; ξ^* and $\Sigma^*(u)$ are to be calculated similarly to the corresponding quantities without asterisk but neglecting the potential and the resonance scattering of the absorber.

Solving numerically the equations (2), (3), or (4), one simultaneously computes the absorption rate. It is, in the lethargy interval between u_1 and u_2,

$$A_{12} = \int_{u_1}^{u_2} \frac{\Sigma_a(u)}{\Sigma(u)} F(u) \, du. \quad (5)$$

In the case of NRIA approximation, $\Sigma(u)$ is to be replaced by $\Sigma^*(u)$. The resonance escape probability for this lethargy interval is

$$p(u_1 \to u_2) = A_{12}/q(u_1), \quad (6)$$

where the slowing down density at the lethargy u_1 is

$$q(u_1) = F(u_1) - \sum_n \frac{\alpha_n}{1 - \alpha_n} \int_{u_1 - \epsilon_1}^{u_1} \frac{\Sigma_{sn}(u')}{\Sigma(u')} F(u') \, du'. \quad (7)$$

In the case of the NRIA approximation again the potential and the resonance scattering of the absorber are to be neglected. If $F(u) = $ const. for $u \leqq u_1$ (i.e., $1/E$ spectrum) one obtains $q(u_1) = \bar{\xi}F(u_1)$, or $q(u_1) = \xi^*F(u_1)$ for the

NRIA approximation. Using the computed resonance escape probability (6), the effective resonance integral $I(u_1 \to u_2)$ is defined by

$$I(u_1 \to u_2) \equiv -\bar{\xi}\sigma_p \ln p(u_1 \to u_2). \quad (8)$$

In this definition, σ_p is the potential scattering cross section per atom of absorber. As we consequently use in all calculations the same definition (8), in the case of the NRIA approximation there is a slight dependence of the effective resonance integral on the moderator properties.

In the integration of the equations (2), (3), or (4) one saves much machine time by appropriate choice of the lethargy step sizes. It is desirable to choose the lethargy step sizes in such a manner that the error is the same in each lethargy step at the computation of the absorption (5). In this way, the important resonance with high contributions to the total absorption will be calculated more accurately. As a rough estimate of the effective resonance integral of a single resonance, which is useful at the determination of the lethargy step sizes, we use the expression

$$I' = I_\infty \frac{1}{\sqrt{1 + \sigma_0/\sigma_p}}$$
$$\cdot \left[1 + \frac{0.18 \, \theta^{0.43}}{1 + 0.134 \left| \frac{\sigma_0}{2.5 \, \sigma_p \sqrt{\theta}} - \frac{2.5 \, \sigma_p \sqrt{\theta}}{\sigma_0} \right|} \right]. \quad (9)$$

I_∞ is the effective resonance integral for infinite dilution; σ_0 is the total resonance cross section at the resonance energy E_0 and $T = 0$; $\theta = 4kTE_0/A\Gamma^2$, k is Boltzmann's constant; T is the absolute temperature; A is the atomic mass number of the absorber and Γ is the total width of the level.

At the time when the code was written our best approximation of the Doppler broadened line shape was

$$\psi(x, \theta) \approx \frac{1}{1 + x^2} \left[3.3(1 - e^{-x^2/4\theta}) - 2.3 \frac{x^2}{x^2 + 2.6128\,\theta} \right.$$
$$\left. - \frac{1.4436}{1 + \sqrt{2.6128\,\theta}} [0.48 \, e^{-0.172x^2/\theta} + 0.52 \, e^{-0.348x^2/\theta}] \right]. \quad (10)$$

Results of some calculations of the effective resonance integral of U^{238} for the 55 resolved resonances below 1000 ev are given in Table I. The resonance parameters are taken from ref. 2. Each resonance is calculated as a single one without interference with other resonances. The ratio of the moderator to uranium atoms is determined by the

TABLE I

RESONANCE INTEGRALS OF U^{238} FOR RESOLVED RESONANCES BELOW 1000 EV AND SEVERAL MODERATORS (IN BARNS)

| σ_p (barn)	T (°K)	Without interference scattering					With interference scattering				
		NR	NR-NRIA		Exact		NR	NR-NRIA		Exact	
			H₂O	C	H₂O	D₂O		H₂O	C	H₂O	D₂O
	300	10.19	10.19	11.03	9.64	9.50	10.55	10.23	10.63	10.33	10.19
30	1000	11.28	11.33	12.27	10.76	10.58	11.54	11.35	11.50	11.19	11.03
	(β)	(0.0076)	(0.0078)	(0.0079)	(0.0079)	(0.0079)	(0.0065)	(0.0076)	(0.0057)	(0.0058)	(0.0058)
	300	35.63	41.95	40.28	38.55	38.36	35.69	41.97	39.80	38.60	38.26
300	1000	42.92	51.17	50.04	47.40	47.12	42.95	51.17	49.40	47.40	46.97
	(β)	(0.0143)	(0.0154)	(0.0169)	(0.0161)	(0.0160)	(0.0142)	(0.0154)	(0.0169)	(0.0160)	(0.0159)
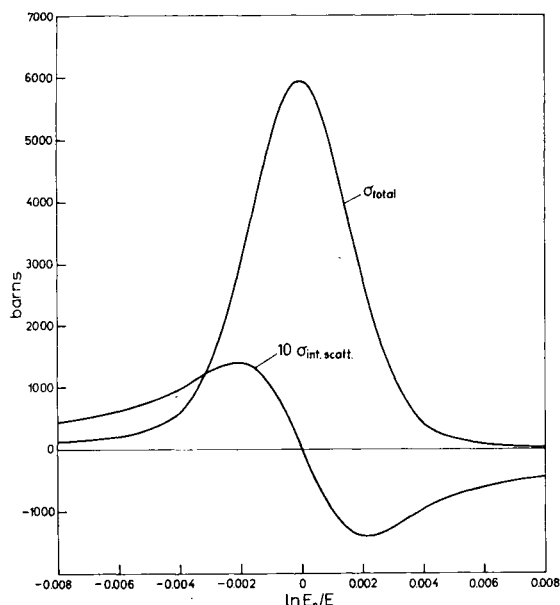
FIG. 1. Variation of total and interference scattering in the U^{238} resonance near the resonance energy $E_0 = 102.8$ ev for $T = 300°K$.

potential scattering cross section per uranium atom σ_p, as the potential scattering cross section of the uranium atom itself is equal to 10 barns. In all three equations considered, (2), (3), and (4) (exact, NR, and NRIA), the same numerical integration with the same lethargy steps is used. Taking the calculated values for two temperatures, the temperature coefficient of the effective resonance integral β

$$I(T) = I(T_0)[1 + \beta(\sqrt{T} - \sqrt{T_0})]. \qquad (11)$$

The results in the columns denoted by NR-NRIA are obtained calculating the six broadest resonances (6.68, 21.0, 36.8, 66.3, 102.8, and 190.0 ev) according to the NRIA approximation and the remaining 49 resonances according to the NR approximation. Comparing the results of the NR and of the NR-NRIA approximations with the exact results for H_2O, it seems that the introduction of the NRIA approximation does not decrease very much the error of the pure NR approximation.

Comparing the exact resonance integrals for three different moderators at the same potential scattering cross section per uranium atom σ_p, one sees noticeable differences between them. As indicated in eq. (2), these differences are caused by different maximal lethargy losses per collision on moderator atoms $\epsilon_n = \ln 1/\alpha_n$ and by the different values of the scattering cross sections Σ_{sn} of the elements of light and heavy water. The effects are somewhat smaller if one also considers the interference between the potential and the resonance scattering. It is interesting to note that the exact temperature coefficient practically does not vary with the moderator properties but is noticeably affected by the interference scattering at small values of the potential scattering per uranium atom σ_p. As the increase of the resonance integral caused by the interference scattering is stronger at lower temperature, the temperature coefficient becomes smaller if one also takes into consideration the interference scattering. (Illustration for a resonance with relatively large interference scattering is given in Fig. 1.) The exact calculation compared with the NR approxima-

tion requires a reasonable increase of the machine time depending on the number of elements in the mixture.

We wish to express our gratitude to Mr. H. J. Siegert for carrying out calculations on a digital computer.

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Velocity Dependent Neutron Transport Theory with High Energy Sources

In a recent paper Conkie (1) described a method of finding analytic approximate solutions to the Boltzmann equation dependent on both position and energy. Conkie's work was confined to the problem of thermal neutrons. In the present paper we wish to extend the method to the case where high energy neutrons are produced in the moderator. We use plane geometry and assume the moderator to be a slab occupying $|x| \leqq x_0$ and surrounded by vacuum. Heavy gas moderation is further assumed, with no capture, and with the neutron sources isotropic and monoenergetic and constantly distributed over the moderator.

The neutron velocities thus cover a broad interval. As in the paper by Conkie we are mainly interested here in obtaining the thermal neutron distribution. We therefore calculate the slowing down density at some velocity v_0 close to but greater than $\sqrt{2kT}$, and this density is then used to give the source term for the thermal region. The slowing down solution can be found in different ways. We use the well-known Greuling-Goertzel method (2). The source velocity is chosen to be $10^4 \sqrt{2kT}$ and the moderator mass M is put equal to 3.6 to represent D_2O (3). The reason is that we wish to perform the calculations for a moderator mass different from 1 but not too big. Details of the calculations can be found in ref. 4. We find the distribution of neutrons at $v_0 = 3.355 \sqrt{2kT}$ to be

$$N_S(x, v, \mu) = [F_0(x) + 3\mu F_1(x)]/2v_0^2 \qquad (1)$$

where

$$F_0(x) = 2.154 - 2 \exp(-\lambda x_0)$$

$$[0.6773x \sinh x - (2.140 + 0.6773x_0) \cosh \lambda x]$$

$$- 22.36 \exp(-\lambda_1 x_0) \cosh \lambda_1 x + 2 \exp(-\lambda_2 x_0) \qquad (2)$$

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