rent is given by $-D(\partial\phi/\partial r)$, Tait's equations give

$$r_1 D_1 \left(\frac{\partial\phi}{\partial r}\right)_1 = r_2 D_2 \left(\frac{\partial\phi}{\partial r}\right)_2 \tag{5}$$

$$\frac{1}{2}\phi(r_1) + D_1\left(\frac{\partial\phi}{\partial r}\right)_1 = \frac{1}{2}\phi(r_2) + \left(\frac{2}{\pi}\right) D_2\left(\frac{\partial\phi}{\partial r}\right)_2 \left\{\frac{r_2}{r_1}\sin^{-1}\left(\frac{r_1}{r_2}\right) + \left[1 - \left(\frac{r_1}{r_2}\right)^2\right]^{1/2}\right\}. \tag{6}$$

Note that diffusion theory and P-1 give the same condition on the current; cf. Eqs. (5) and (3). Using Eq. (5), Eq. (6) becomes

$$\phi(r_1) - \phi(r_2) = -2D_1\left(\frac{\partial\phi}{\partial r}\right)_1 \cdot \left\{1 - \frac{2}{\pi}\left[\sin^{-1}\left(\frac{r_1}{r_2}\right) + \frac{r_1}{r_2}\sqrt{1 - \left(\frac{r_1}{r_2}\right)^2}\right]\right\}. \tag{7}$$

Equating Eqs. (7) and (4) and solving the resulting expression for the void diffusion coefficient gives

$$D/r_1 = \frac{\ln(r_2/r_1)}{2\{1 - (2/\pi)[\sin^{-1}(r_1/r_2) + (r_1/r_2)\sqrt{1 - (r_1/r_2)^2}]\}}. \tag{8}$$

Therefore, by using the above expression for the void diffusion coefficient in diffusion theory calculations the P-1 boundary conditions for an annular void region are preserved. Note that D is purely a function of geometry and hence group independent. For $(r_1/r_2) \ll 1$, Eq. (8) becomes

$$D/r_1 = \frac{\ln(r_2/r_1)}{2[1 - (4/\pi)(r_1/r_2)]} \tag{9}$$

and hence $(D/r_1) = \frac{1}{2}\ln(r_2/r_1)$ as $(r_1/r_2) \to 0$. Table I gives values of (D/r_1) as a function of the ratio (r_1/r_2). It should be noted that the above treatment is also valid for $\Sigma_a = 0$.

An analogous expression for the void diffusion coefficient for a spherical annular region can similarly be derived. Extending the material as given by Davison (*2*), the annular void boundary conditions for the spherical case are

$$r_1^2\left(\frac{\partial\phi}{\partial r}\right)_1 = r_2^2\left(\frac{\partial\phi}{\partial r}\right)_2 \tag{10}$$

$$\frac{1}{2}\phi(r_1) + D_1(\partial\phi/\partial r)_1 = \frac{1}{2}\phi(r_2) + D_2\left(\frac{\partial\phi}{\partial r}\right)_2\left\{1 - \left[1 - \left(\frac{r_1}{r_2}\right)^2\right]^{3/2}\right\}. \tag{11}$$

Following the same procedure as in the cylindrical case gives

$$\frac{D}{r_1} = \frac{1}{2}\left[1 - \frac{r_1}{r_2}\right]\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]^{-3/2} \tag{12}$$

for the void diffusion coefficient. Again the above value of D in a diffusion theory calculation preserves the conditions expressed by Eqs. (10) and (11).

As is well known, a void region in plane geometry can simply be neglected since the void does not contribute to the optical thickness. However, it is interesting to consider the plane case in light of the above formulation. For the plane case, the equation corresponding to Eq. (4) is given by

$$\phi(x_1) - \phi(x_2) = \frac{D_1}{D}\left(\frac{\partial\phi}{\partial x}\right)_1 (x_1 - x_2). \tag{13}$$

To find the value of D for the plane case, we make the substitution $r_2 = (r_1 + t)$ in Eq. (12) for $t \ll r_1$, r_2 and consider $r_1 \to \infty$; this gives

$$D = (r_1^3/2^5 t)^{1/2}, \tag{14}$$

hardly a surprising result since as $r_1 \to \infty$ the spherical case approaches the plane case with a void region of thickness t and hence the right hand side of Eq. (13) vanishes giving the expected result that $\phi(x_1) = \phi(x_2)$.

It should be emphasized that for the cylindrical and slab cases, the treatment is restricted to infinite cylinders and infinite planes; i.e., no end leakage. The end leakage is a separate problem.

TABLE I

VALUES OF D/r_1

r_1/r_2	D/r_1	
	Cylinder	Sphere
0.99	4.2427	1.7811
0.9	1.4089	0.60373
0.8	1.0719	0.46296
0.7	0.94799	0.41185
0.6	0.89696	0.39063
0.5	0.88637	0.38490
0.4	0.90788	0.38967

REFERENCES

1. J. H. TAIT, *Proc. 1st Intern. Conf. Peaceful Uses Atomic Energy, Geneva, 1955*, P/433.
2. B. DAVISON, "Neutron Transport Theory," chap. XI. Oxford Univ. Press, 1957.

E. GARELIS

Vallecitos Atomic Laboratory
General Electric Company
Pleasanton, California
Received November 27, 1961

On the Influence of Pressure on Boiling Water Reactor Dynamic Behavior at Atmospheric Pressure

Due to the present importance of understanding reactor kinetics, it is essential to be as rigorous as possible in the mathematical modeling of reactor problems. Regrettably, such has not been the case in most analyses in this field. On the other hand very great care has been taken with the analysis of the hydrodynamic fields. Overwhelming mathematical detail has been introduced, with the result that integration can be done only on a computer. However, the basic equation of reactor kinetics has been modeled incorrectly.

To point up the practical aspects and consequences of this deficiency, it will be shown that pressure oscillations in a reactor can arise in a way which would not be found in the analysis, complete though it is, done by Fleck (*1*). In his paper, he assumed the reactor kinetic equation (Eq. 26) with the last term on the right in brackets omitted

$$\frac{d\phi}{dt} = \left(\frac{k_{ex} - \beta}{l}\right)\phi + \sum_i \lambda_i C_i + \left[\frac{\partial \phi}{\partial V}\dot{V}\right] \tag{1}$$

The rigorous necessity for terms of this kind has previously been demonstrated (*2*). A careful analysis of the basis of the usual reactor kinetics equation shows that it is valid if and only if the composition (including the exact position of all parts) of the reactor does not change with time. A detailed and careful derivation of the revised reactor kinetic equation in terms of reactor cross sections, including physical interpretation of the added terms, has been given in an unpublished report (*3*). For convenience, the last term in (1) can be rewritten in terms of f_{av} directly as:

$$\frac{\partial \ln\phi}{\partial f_{av}}\left[\frac{df_{av}}{dt}\right]\phi \tag{2}$$

As in ref. *1*, let

$$k_{ex} = \epsilon - \alpha f_{av} \tag{3}$$

In order to investigate the possibility of oscillations in (1), we make the standard assumption that f_{av} may be split into a constant term and a periodic term:

$$f_{av} = f_0 + \hat{f}\exp\left[(\gamma + i\omega)(t - t_0)\right] \tag{4}$$

and that

$$\sum_i \lambda_i C_i/\phi\,|_t = \sum_i \lambda_i C_i/\phi\,|_{t=t_0} + (t - t_0)\frac{d}{dt}\left[\sum_i \lambda_i C_i/\phi\right]$$

We now assume that the reactor is operating near the steady state point so that

$$\left.\frac{(\epsilon - \alpha f_0 - \beta)}{l} + \frac{\sum_i \lambda_i C_i}{\phi}\right|_{t=t_0} = 0 \tag{5}$$

Equations (2)–(5) are now substituted in Eq. (1) to give

$$\frac{d\ln\phi}{dt} = -\frac{\alpha\hat{f}\exp\left[(\gamma + i\omega)(t - t_0)\right]}{l} + \left[\frac{\partial \ln\phi}{\partial f_{av}}\right]\frac{df_{av}}{dt} + (t - t_0)\frac{d}{dt}\left[\frac{\sum_i \lambda_i C_i}{\phi}\right] \tag{6}$$

As explained in ref. *2*, the expression

$$\frac{k_{ex} - \beta}{l}$$

in Eq. (1) must be considered the partial of $\ln\phi$ with respect to time, provided $(k_{ex} - \beta)/l$ is large compared to $\sum_i \lambda_i C_i/\phi$. It follows from the rules of partial differentiation (*4*) that if $\ln\phi$, a function of the dependent variable, f_{av}, and the independent variable, t, is continuous with continuous first and second partial derivatives then:

$$\frac{\partial}{\partial t}\left[\frac{\partial \ln\phi}{\partial f_{av}}\right] = \frac{\partial}{\partial f_{av}}\left[\frac{\partial \ln\phi}{\partial t}\right] = \frac{\partial}{\partial f_{av}}\left[\frac{k_{ex} - \beta}{l}\right] = -\frac{\alpha}{l} \tag{7}$$

This equation may be integrated to obtain

$$\frac{\partial \ln\phi}{\partial f_{av}} = -\frac{\alpha}{l}(t - t_0) + \left.\frac{\partial \ln\phi}{\partial f_{av}}\right|_{t=t_0} \tag{8}$$

Now, substitute (8) and the value of df_{av}/dt from (4) in (6), and it is found that, if $\partial\ln\phi/\partial f_{av}\,|_{t=t_0}$ is small, then

$$\frac{d}{dt}(\ln\phi) = -\frac{\alpha\hat{f}}{l}\exp\left[(\gamma + i\omega)(t - t_0)\right] \cdot\left[1 + (t - t_0)(\gamma + i\omega)\right] + (t - t_0)\frac{d}{dt}\left[\sum_i \lambda_i C_i/\phi\right] \tag{9}$$

The last term on the right is small in certain circumstances so that Eq. (9) may be integrated. The conditions are found from the following simultaneous equations: The first equation is the expansion of the derivative. The second equation is the standard expression for the rate of change of C_i.

$$\frac{d}{dt}\left[\frac{\sum_i \lambda_i C_i}{\phi}\right] = \frac{1}{\phi^2}\left[\frac{d}{dt}\left(\sum_i \lambda_i C_i\right) - \frac{d\phi}{dt}\left(\sum_i \lambda_i C_i\right)\right] \tag{10}$$

$$\frac{dC_i}{dt} = -\lambda_i C_i + \frac{\beta a_i}{l}\phi \tag{11}$$

Equation (10) becomes the following upon substitution of Eq. (11):

$$\frac{d}{dt}\left[\frac{\sum_i \lambda_i C_i}{\phi}\right] = \sum_i \lambda_i\left[\frac{a_i\beta}{l\phi} - \frac{C_i}{\phi}\left(\frac{\lambda_i}{\phi} + \frac{d\ln\phi}{dt}\right)\right] \tag{12}$$

For the product $(t - t_0)d/dt[\sum_i \lambda_i C_i/\phi]$ to be small, it is sufficient that the following is true:

$$(t - t_0)\lambda_i\left[\frac{a_i\beta}{l\phi} - \frac{C_i}{\phi}\left[\frac{\lambda_i}{\phi} + \frac{d\ln\phi}{dt}\right]\right] \cong 0 \tag{13}$$

The first two terms on the left, near the steady state operation of the reactor, cancel each other out (because Eq. (11) is small) so that the term of largest value is the third.

We compare the third term with the sum of all terms on the right of Eq. (9) in order to determine the conditions under which it is a good approximation to neglect it:

$$\frac{(-(t - t_0)C_i\lambda_i/\phi)(d\ln\phi/dt)}{d\ln\phi/dt} = -\frac{(t - t_0)C_i\lambda_i}{\phi}$$

This ratio is small only when $(t - t_0)$ is small. If a long time were involved, the third term may become large and thereby tend to counteract a positive rate of change of $\ln\phi$. As a result, it is found that Eq. (9) may be integrated neglecting the third term only if $(t - t_0)$ is small compared to $\phi/C_i\lambda_i$, and if $(k_{ex} - \beta)/l$ is large compared to $\sum_i \lambda_i C_i/\phi$.

It is appropriate to digress for a moment and return to Eq. (9) to make a second comparison. This digression is to show the relative influence of the term which comes from the proper form of Eq. (1) in Eq. (9). The terms which are to be compared for this purpose are the last term in Eq. (13) and the second term in Eq. (9) (which comes from the proper form of Eq. (1)). The ratio of these two terms is:

$$-\frac{(t - t_0)C_i\lambda_i}{\phi}\frac{d\ln\phi}{dt} \cdot\left[(t - t_0)(\gamma + i\omega)\left(-\frac{\alpha\hat{f}}{l}\right)\exp\left[(\gamma + i\omega)(t - t_0)\right]\right]^{-1} \cong \frac{C_i\lambda_i}{\phi(\gamma + i\omega)}$$

From this ratio it is seen that if $(\gamma^2 + \omega^2)^{1/2}$ is large compared to $C_i\lambda_i/\phi$, then in Eq. (9) the second term is large compared to the third term. It is concluded from this ratio that, in any reactor transient experiment, the time-dependent portions of the average void fraction must be measured in such a way so as to determine the value of both the e-folding coefficient, γ, and ω, the angular frequency, to the same precision as $C_i\lambda_i/\phi$.

Returning now to the integration of Eq. (9) we may neglect the third term on the right, subject to the restrictions found, i.e., for small time of departure from the steady state, and integrate the remaining terms to find

$$\phi = \phi_0 \exp\left\{-\frac{\alpha\hat{f}(t - t_0)}{l} \exp\left[(\gamma + i\omega)(t - t_0)\right]\right\} \quad (14)$$

If Eq. (6) were integrated without the second or extra term, then ϕ would be given as follows:

$$\phi = \phi_0 \exp\left\{-\frac{\alpha\hat{f}}{l}\left(\frac{\gamma - i\omega}{\gamma^2 + \omega^2}\right)(\exp\left[(\gamma + i\omega)(t - t_0)\right] - 1)\right\} \quad (15)$$

The comparison of Eqs. (14) and (15), and the comparison of the form of Eq. (9) with and without the term $(t - t_0)(\gamma + i\omega)$ in the large bracket shows that Eq. (14) predicts a much more rapid change of power than does Eq. (13) as soon as the time, $(t - t_0)$ becomes greater than $(\gamma^2 + \omega^2)^{-1/2}$.

Equation (14) shows that either divergent or convergent oscillations are possible, although Fleck claimed divergent oscillations would not occur. To determine the exact coupling between f_{av} and ϕ for any reactor requires a more complete study than the present, such as that done by Fleck (*1*). Even without such a complete study, the following conclusions can be noted from Eq. (14).

1. If α is positive, the power oscillations are damped (no matter what the value of γ).

2. If α is negative, the power oscillations grow and reactor runaway may occur. In such a case (which would correspond to voids in an overmoderated region of a reactor), the oscillations grow until the growth is stopped due to α reversing its sign and becoming positive. For α to do this may require the expulsion of moderator (all other properties assumed unchanged).

3. If the term (2) were not introduced in (1), then there would be a much smaller feedback in Eq. (1) as shown in both Eqs. (9) and (14).

4. The most important conclusion is that the standard reactor kinetic equation (1) for the reactor flux which does not contain the rate of change terms (as explained in ref. *2*) can give false results and conclusions based on it are not to be trusted. This conclusion must be considered valid for reactors of all types unless the extra terms are proved by both theory and experiment to be negligible.

LIST OF SYMBOLS

a_i = fraction of all delayed neutrons which belong to the ith group of delayed neutrons
C_i = number density of neutron precursors of the ith group
V = void volume averaged over reactor core
f_{av} = void fraction averaged over whole core
f_0 = constant portion of the void fraction averaged over whole core
$\hat{f}$ = amplitude of fluctuating component of the void fraction
k_{ex} = excess multiplication factor
l = mean lifetime of neutrons in unperturbed reactor
ϕ = reactor neutron flux
$-\alpha$ = void fraction coefficient of reactivity
β = fraction of fission neutrons which are delayed
γ = "e-folding" coefficient for fluctuating component of average void fraction (a positive quantity, the inverse of the "e-folding" time)
ϵ = excess multiplication factor with no void
λ_i = decay constant for neutron precursors of the ith group
ω = angular frequency of the periodic fluctuation of the average void fraction

REFERENCES

1. Joseph A. Fleck, Jr., *Nuclear Sci. and Eng.* **9**, 271–231 (1961).
2. Richard L. Moore, *Nuclear Sci. and Eng.*, **6**, 157 (1959).
3. Richard L. Moore, Relation between density fluctuations and reactor power fluctuations. R. L. Moore Consultants, Los Angeles, 17 August 1960. Revised 30 January 1961, *Nuclear Sci. Abstr.* **15:** 12470 (1961).
4. E. Goursat and E. R. Hedrick, "A Course in Mathematical Analysis", Vol. I, pp. 13, 25. Ginn, Boston, 1904.

Richard L. Moore
R. L. Moore, Consultants
Los Angeles, California
Received April 5, 1961
Revised January 24, 1962