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Letters to the Editors

Space-Time Burnout of an Absorbing Slab

Consider a nonscattering absorbing slab extending from $x = 0$ to $x = D$. Assume a constant current of neutrons of strength ϕ_0 entering at $x = 0$ normal to the surface, and starting at time zero. Let the slab have a single absorbing species of microscopic cross section σ barns and initial number density N_0 nuclei per barn-cm. Then, at any time t and depth x, the neutron current and absorber number density satisfy the following simultaneous integral equations:

$$N(x, t) = N_0 \exp\left[-\sigma \int_0^t \phi(x, t')\, dt'\right] \tag{1}$$

$$\phi(x, t) = \phi_0 \exp\left[-\sigma \int_0^x N(x', t)\, dx'\right]. \tag{2}$$

These equations can easily be transformed to a pair of simultaneous differential equations by changing to a new set of variables:

$$u(x) = \sigma nvt \text{ at depth } x = \sigma \int_0^t \phi(x, t')\, dt' \tag{3}$$

$$v(t) = \text{depth in mean free paths} = \sigma \int_0^x N(x', t)\, dx'. \tag{4}$$

Equations (1) and (2) become

$$\partial v/\partial x = \sigma N_0 e^{-u} \tag{5}$$

$$\partial u/\partial t = \sigma \phi_0 e^{-v}. \tag{6}$$

A solution is easily found in the following form[1]

$$u = \ln\,[1 + e^{\sigma\phi_0 t - \sigma N_0 x} - e^{-\sigma N_0 x}] \tag{7}$$

$$v = \ln\,[1 + e^{\sigma N_0 x - \sigma\phi_0 t} - e^{-\sigma\phi_0 t}]. \tag{8}$$

The ϕ and N can be obtained by differentiation but often u and v are themselves more valuable. For example, $v(D)$ is the mean free path depth of the slab at any time.

A similar case of greater interest is simply to solve the same problem when the slab is subjected to a normally incident current from both sides. In this case let the total thickness of the slab be $2x$, and examine the conditions which apply at the center

$$\phi(x, t) = 2\phi_0 \exp\left[-\sigma \int_0^x N(x', t)\, dx'\right] \tag{9}$$

$$N(x, t) = N_0 \exp\left[-\sigma \int_0^t \phi(x, t')\, dt'\right]. \tag{10}$$

The solution proceeds in the same manner as above, yielding the following value for the slab depth (in mean free paths) as a function of time:

$$2v(x, t) = 2 \ln\,[1 + e^{N_0\sigma x - 2\phi_0\sigma t} - e^{-2\phi_0\sigma t}].$$

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[1] This solution proceeds from the fact that $\partial^2 u/(\partial x \partial t)$ and $\partial^2 v/(\partial x \partial t)$ are equal, so that $u + h(x) = v + g(t)$. The solution follows upon insertion into Eqs. (5) and (6) and use of boundary conditions.

Concerning the Theory of Control Sheets

In a recent paper (*1*), Wolfe derived a critical condition for a plane symmetric reactor with plane control sheets inserted, under the conditions that

$$\delta \gg \min\,(L, \sqrt{\tau}) \tag{1}$$

where δ is the spacing between sheets, and L, τ are the thermal diffusion length and age in the core material. In particular, a critical equation of the form

$$(\sin \mu\delta, \cos \mu\delta)(\alpha\lambda_1^N V_1 + \beta\lambda_2^N V_2) = 0 \tag{2}$$

has been given for N equally spaced sheets, where α, β, λ_1, λ_2 are functions of the material properties, and V_1, V_2 are vector functions of these properties.

Equation (2) was derived from the condition

$$(\sin \mu\delta, \cos \mu\delta) Q^N \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \tag{3}$$

where

$$Q = \begin{pmatrix} \cos \mu\delta + R \sin \mu\delta & -\sin \mu\delta + R \cos \mu\delta \\ \sin \mu\delta & \cos \mu\delta \end{pmatrix} \tag{4}$$

and R is a function of material properties.

In the following we will show how the critical equation, Eq. (2), can be considerably simplified by working from Eqs. (3) and (4) in a somewhat different manner than was done in ref. *1*.

It was shown in ref. *1* that the eigenvectors V_1, V_2 of Q are

$$V_1 = \begin{pmatrix} S - RC \\ RS/2 - \sqrt{T^2 - 1} \end{pmatrix} \tag{5}$$

$$V_2 = \begin{pmatrix} S - RC \\ RS/2 + \sqrt{T^2 - 1} \end{pmatrix} \tag{6}$$

where we have abbreviated

$$S = \sin \mu\delta; \qquad C = \cos \mu\delta \tag{7}$$

$$T = C + (RS/2) \tag{8}$$

with eigenvalues

$$\lambda_{\substack{1\\2}} = T \pm \sqrt{T^2 - 1}. \tag{9}$$

That being the case, we have

$$QP = P\Lambda \tag{10}$$

where

$$P = \begin{pmatrix} S - RC & S - RC \\ RS/2 - \sqrt{T^2 - 1} & RS/2 + \sqrt{T^2 - 1} \end{pmatrix} \tag{11}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{12}$$

or

$$Q = P\Lambda P^{-1}. \tag{13}$$

Thus,

$$Q^N = P\Lambda^N P^{-1} \tag{14}$$

or, more explicitly,

$$Q^N = \frac{1}{\Delta} \begin{pmatrix} S - RC & S - RC \\ \frac{RS}{2} - \sqrt{T^2 - 1} & \frac{RS}{2} + \sqrt{T^2 - 1} \end{pmatrix} \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} \begin{pmatrix} \frac{RS}{2} + \sqrt{T^2 - 1} & CR - S \\ \sqrt{T^2 - 1} - \frac{RS}{2} & S - RC \end{pmatrix} \tag{15}$$

where

$$\Delta = 2\sqrt{T^2 - 1}\,(S - RC). \tag{16}$$

By direct calculation from Eqs. (3) and (15), we find the surprisingly simple critical equation

$$(S, C)Q^N \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{S}{2\sqrt{T^2 - 1}} \{\lambda_1^{N+1} - \lambda_2^{N+1}\} = 0. \tag{17}$$

By inspection, one can verify that the only admissible solutions of Eq. (17) are those for which

$$(\lambda_1/\lambda_2)^{N+1} = 1. \tag{18}$$

If we write $T = \cos \psi$ in Eq. (9), and substitute in Eq. (18)

$$e^{2i(N+1)\psi} = 1 \tag{19}$$

that is, the critical values of $\psi = \cos^{-1} T$ are

$$\psi_j = j\pi/(N + 1) \quad (j = 1, 2, \cdots, 2N + 1) \tag{20}$$

and the critical equation, from Eqs. (8) and (20), takes the simple form

$$\cos \mu\delta + \frac{R}{2} \sin \mu\delta = \cos \frac{j\pi}{N + 1} \qquad (j = 1, 2, \cdots, 2N + 1) \tag{21}$$

Equation (21) is understood to be solved for each j, and the smallest positive root for k so obtained is the desired eigenvalue.

It is of interest to note that Eq. (21) can be solved explicitly for the critical spacing δ, in the form

$$\delta_{\text{crit}} = \frac{1}{\mu} \cos^{-1} \left\{ \frac{\cos [j\pi/(N + 1)] \pm \frac{1}{2}R\sqrt{\sin^2[j\pi/(N + 1)] + R^2/4}}{1 + R^2/4} \right\} \tag{22}$$

which is to be interpreted in a manner similar to Eq. (21).

REFERENCE

1. B. Wolfe, "General Theory of Control Sheets," *Nuclear Sci. and Eng.* **4**, 635–648 (1958).

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A Simple Treatment for Effective Resonance Absorption Cross Sections in Dense Lattices

It has recently been shown by Chernick *et al.* (*1, 2*) that effective resonance absorption cross sections can be computed with the same expressions for both homogeneous mixtures of absorber and moderator and also for isolated[1] lumps of absorber in moderator. This result was obtained by making for the isolated lump case, the so-called Wigner or canonical approximation to the neutron escape probability from a lump. Let S denote lump area, V_0 lump volume, V_1 moderator volume per lump, Σ_0 macroscopic cross section in lump, and Σ_1 moderator cross section. In this notation, it was found that the quantity $S/4V_0 = s_0$ plays the same role for the heterogeneous case that the moderator cross section per absorber atom ($\Sigma_1 V_1/V_0$) plays in the homogeneous case. The quantity s_0 was interpreted as a pseudo-cross section representing escape from the lump (*2*).

For the case of dense lattices with closely spaced lumps, it has been customary to apply Dancoff corrections (*3*) to the isolated lump case. This is frequently a quite complicated procedure. It is the purpose of this note to indicate how the canonical treatment may be generalized to the case of closely spaced lumps and to obtain a transition between the isolated lump and homogeneous cases. The result of such a generalization is very simple; namely, in general the quantity s_0 is to be replaced (in all isolated lump expressions) by τ_0, where

$$\tau_0 = \frac{s_0 \Sigma_1}{\Sigma_1 + s_0 (V_0/V_1)}. \tag{1}$$

In the following, we shall first give a heuristic justification of this recipe and then note some of its desirable properties.

We assume, as usual, that neutrons arrive at any energy E uniformly in space within either the absorber lump

[1] By isolated we mean that separation between lumps is large compared to a moderator mean free path.