Letter to the Editor

A Direct Calculation of the Time-Dependent Spatial Moments of the One-Speed Neutron Transport Model

Barnett,[1] in a recent Letter to the Editor in reply to comments made by Cohen,[2] asked if time-dependent spatial moments could be obtained directly from the Case and Zweifel[3] solution to the one-speed transport equation in plane geometry. In this Letter, I will show that from this solution spatial moments in plane geometry as well as in spherical and cylindrical geometry can be found. Some numerical results facilitating the moments calculation will also be given.

In Ref. 4, I have shown that the Case and Zweifel solution for an isotropic plane source[5] and isotropic scattering is equivalent to

$$\phi^{pl}(x,\mu,t) = \frac{\exp(-vt)}{2t}\left[\delta(\mu-\eta) + \sum_{n=1}^{\infty}\left(\frac{cvt}{2}\right)^n T_n^{n-1}(\mu,\eta)\right] H(1-|\eta|) \ , \qquad (1)$$

where

$$T_n^{n-1}(\mu,\eta) \equiv -\frac{1}{2\pi i}\,\frac{1}{(n-1)!}\int_\eta^1 (z'-\eta)^{n-1} M_n(\mu,z')\,dz'$$

$$M_n(\mu,z') \equiv \left\{\left[P\frac{1}{z'-\mu} - i\pi\delta(z'-\mu)\right]\left[\ln\left|\frac{1+z'}{1-z'}\right| - i\pi\right]^n - \left[P\frac{1}{z'-\mu} + i\pi\delta(z'-\mu)\right]\left[\ln\left|\frac{1+z'}{1-z'}\right| + i\pi\right]^n\right\}$$

$$\eta \equiv x/vt$$

$$H(u) \equiv \begin{cases} 0 \ , & u \le 0 \\ 1 \ , & u > 0 \end{cases} \ .$$

The n'th term in Eq. (1) represents the angular flux of neutrons $\phi_n^{pl}(x,\mu,t)$ that have undergone n collisions.

Integrating Eq. (1) over μ gives the scalar flux,

$$\phi(x,t) = \frac{\exp(-vt)}{2t}\left[1 + \sum_{n=1}^{\infty}\left(\frac{cvt}{2}\right)^n (n+1)Q_{n+1}^{n-1}(\eta)\right] \times H(1-|\eta|) \ , \qquad (2)$$

with

$$Q_k^p(\eta) \equiv -\frac{1}{2\pi i}\,\frac{p+1}{k!}\int_\eta^1 (z'-\eta)^p M_k(z')\,dz'$$

$$M_k(z') \equiv \left[\ln\left|\frac{1+z'}{1-z'}\right| - i\pi\right]^k - \left[\ln\left|\frac{1+z'}{1-z'}\right| + i\pi\right]^k \ .$$

The method of evaluating $\phi^{pl}(x,t)$ is outlined in Ref. 6. The ν'th time-dependent spatial moment in plane geometry is therefore

$$G_\nu^{pl}(t) = \int_{-\infty}^{\infty} x^\nu \phi^{pl}(x,t)\,dx \ ,$$

or since $\phi^{pl}(x,t)$ is even in x,

$$G_{2k}^{pl}(t) = \frac{\exp(-vt)}{t}\,\frac{(vt)^{2k+1}}{2k+1} \times \left[1 + (2k+1)!\sum_{n=1}^{\infty}\left(\frac{cvt}{2}\right)^n \frac{(n+1)!}{(n+2k+1)!} Q_{n+1}^{n+2k}(0)\right] \qquad (3)$$

$$G_{2k+1}^{pl}(t) = 0$$

for $k = 0, 1, \ldots$. In deriving this expression, use was made of the relation of Ref. 4,

$$\int_0^1 (\eta')^m Q_{n+1}^{n-1}(\eta')\,d\eta' = \frac{n!\,m!}{(n+m+1)!} Q_{n+1}^{n+m} \ , \qquad (4)$$

where

$$Q_{n+1}^{n+m} \equiv Q_{n+1}^{n+m}(0) \ .$$

From a recursion relation given in Ref. (6), we have

$$Q_{n+1}^{n+2k} = \frac{2}{n+2k} Q_n^{n-1+2k} + Q_{n+1}^{n+2k-2} \ , \qquad (5a)$$

and it can be shown (see Ref. 4, Chap. 5) that the initial values to be used in Eq. (5a) are

$$Q_{n+1}^n = \frac{2^n}{n!} \ , \quad Q_1^m = 1 \ . \qquad (5b)$$

Thus, Eqs. (6) constitute a simple recursion relation, the solution of which can be inserted into Eq. (3) to yield $G_{2k}^{pl}(t)$.

In spherical geometry, we have[7]

$$\phi^{Sp}(r,t) = -\frac{1}{2\pi r}\,\frac{\partial\phi^{pl}(r,t)}{\partial r} \ ,$$

and the ν'th spherical spatial moment,

$$G_\nu^{Sp}(t) = \int_0^\infty r^\nu \cdot 4\pi r^2 \phi^{Sp}(r,t)\,dr \ ,$$

can be shown to be

$$G_\nu^{Sp}(t) = 2(\nu+1)\int_0^\infty r^\nu \phi^{pl}(r,t)\,dr \ . \qquad (6)$$

[1] C. S. BARNETT, *Nucl. Sci. Eng.*, **52**, 281 (1973).

[2] E. R. COHEN, *Nucl. Sci. Eng.*, **52**, 280 (1973).

[3] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts (1967).

[4] B. D. GANAPOL, "One-Dimensional Time-Dependent Neutron Transport in a Infinite Medium," PhD Thesis, University of California, Berkeley (1971).

[5] Equation (9) on p. 186 of Ref. 3 is halved and integrated over μ_0 for this source.

[6] B. D. GANAPOL and L. M. GROSSMAN, *Nucl. Sci. Eng.*, **52**, 454 (1973).

[7] B. DAVISON and J. B. SYKES, *Neutron Transport Theory*, p. 67, Oxford University Press (1958).

when $\nu = 2k$, it is evident that

$$G_{2k}^{Sp}(t) = (2k+1)\ G_{2k}^{pl}(t) \quad . \tag{7}$$

Specifically for $k = 1$, Q_{n+1}^{n+2} can be found (from the recursion relation) to be

$$Q_{n+1}^{n+2} = \frac{2^n}{3}\ \frac{(n+3)}{n!} \quad ;$$

when this expression is substituted into Eq. (3), we find Barnett's result[8] for the second spatial moment. Performing the integration in Eq. (6) when $\nu = (2k + 1)$ [with the aid of Eq. (4)] leads to the expression

$$G_{2k+1}^{Sp}(t) = \frac{\exp(-vt)}{t}\ (vt)^{2k+2} \times \left[1 + (2k+2)!\sum_{n=1}^{\infty}\left(\frac{cvt}{2}\right)^n \frac{(n+1)!}{(n+2k+2)!} \times Q_{n+1}^{n+2k+1}\right] \tag{8}$$

for the odd spherical spatial moments, where[6]

$$Q_{n+1}^{n+2k+1} = \frac{2}{n+2k+1}\ Q_n^{n+2k} + Q_{n+1}^{n+2k-1} \quad . \tag{9}$$

The starting function for the recursion relation, Q_{n+1}^{n-1}, can be evaluated using the methods given in Ref. (6). The values of Q_{n+1}^{n-1} for $n = 1$ to 50 are given in Table I, thus allowing the implementation of the recursion relation.

The "transport equation approach" of Cohen[2] can give the odd spherical spatial moments [Eq. (6) for $\nu = (2k + 1)$] only if the expression for the plane angular flux distribution (at $x = 0$) is known.

The spatial moments in cylindrical geometry are easily obtained by noting that

$$\phi^{Cy}(\rho,t) = \int_{-\infty}^{\infty} \phi^{Sp}(r,t)\,dz \quad .$$

Then inserting this expression into

$$G_\nu^{Cy}(t) = \int_0^\infty \rho^0 \cdot 2\pi\rho\phi^{Cy}(\rho,t)\,d\rho$$

yields

$$G_\nu^{Cy}(t) = \frac{\sqrt{\pi}}{2}\ \frac{\Gamma\left(\frac{\nu}{2}+1\right)}{\Gamma\left(\frac{\nu}{2}+\frac{3}{2}\right)}\ G_\nu^{Sp}(t) \quad . \tag{10}$$

Kholin[9] has solved the time-dependent transport equation with anisotropic scattering; from his solution it is possible to determine the plane spatial moments for any order of scattering. As the order of scattering increases, the moments become more difficult to calculate and Cohen's method[2] is far easier to use. For other than plane geometry, however, one has no choice but to determine the moments using the full distribution.

TABLE I

Q_{n+1}^{n-1} for $n = 1, 2, \ldots, 50$ to be used in Eq. (9)

n	Q_{n+1}^{n-1}	n	Q_{n+1}^{n-1}
1	1.386294E + 00	26	3.126896E - 20
2	1.127655E + 00	27	2.274539E - 21
3	6.503772E - 01	28	1.596451E - 22
4	2.987723E - 01	29	1.082519E - 23
5	1.861706E - 01	30	7.099588E - 25
6	3.276664E - 02	31	4.508332E - 26
7	8.758002E - 03	32	2.774738E - 27
8	2.064478E - 03	33	1.656774E - 28
9	4.352716E - 04	34	9.605659E - 30
10	8.301056E - 05	35	5.412263E - 31
11	1.445150E - 05	36	2.965947E - 32
12	2.314273E - 06	37	1.582002E - 33
13	3.431152E - 07	38	8.218999E - 35
14	4.735751E - 08	39	4.161907E - 36
15	6.114197E - 09	40	2.055446E - 37
16	7.414964E - 10	41	9.906601E - 39
17	8.478138E - 11	42	4.662307E - 40
18	9.169335E - 12	43	2.143755E - 41
19	9.407947E - 13	44	9.635564E - 43
20	9.181586E - 14	45	4.235712E - 44
21	8.543640E - 15	46	1.821935E - 45
22	7.596486E - 16	47	7.671803E - 47
23	6.466768E - 17	48	3.163827E - 48
24	5.280254E - 18	49	1.278390E - 49
25	4.142289E - 19	50	5.063223E - 51

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[8] C. S. BARNETT, *Nucl. Sci. Eng.*, **50**, 398 (1973).
[9] S. A. KHOLIN, *USSR Com. Math and Math-Phys.*, **4**, 213 (1964).