Letter to the Editor

Transport Equations for Contributons

In a recent paper by Williams and Engle,[1] a concept of spatial channel theory is introduced that utilizes the definition of fictitious particles, called "contributons," to obtain new insights into the physical processes that dominate the transport of neutrons through complex shields. In a parallel development, Gerstl[2] used the same definition of an adjoint-weighted neutron flux $\psi = \phi^+\phi$, called "ψ particles" in Ref. 2 but identical to the contributons of Ref. 1, to reformulate the entire neutron transport problem in terms of ψ. These papers have stirred interest, especially among shielding and radiation transport specialists, because a new concept is introduced that allows the description of neutral particle and radiation transport in terms of contributons. Chilton[3] suggests interpreting contributon transport through a visual representation of the directional values of the contributon current, called contributon lines, to further extract significant insights from this concept. However, a discussion about what to consider a "transport equation" (TE) for contributons has also erupted in Letters to the Editor by Gerstl[4] and Williams.[5] Our continued research in this field has yielded some new information that may help to settle this controversy.

Williams and Engle[1] consider the continuity equation for contributons,

$$\nabla \cdot D(r) = S(r) \ , \tag{1}$$

as a TE for the contributon flux, where

$$D(r) = \int_E \int_{\mathbf{\Omega}} \mathbf{\Omega}\phi(\boldsymbol{\rho})\phi^+(\boldsymbol{\rho})d\mathbf{\Omega}dE \tag{2}$$

is the contributon current and where

$$S(r) = \int_E \int_{\mathbf{\Omega}} [\phi^+(\boldsymbol{\rho}),Q(\boldsymbol{\rho}) - \phi(\boldsymbol{\rho}),Q^+(\boldsymbol{\rho})]d\mathbf{\Omega}dE \tag{3}$$

is the source distribution of contributons. Gerstl[4] considers this characterization inappropriate because Eq. (1) does not allow the calculation of the distribution of contributons without previous knowledge of the forward and adjoint particle flux distributions $\phi(\boldsymbol{\rho})$ and $\phi^+(\boldsymbol{\rho})$. Instead, he claims that an equation for $\psi = \phi\phi^+$ derived in Ref. 2 for a purely absorbing medium,

$$\mu^4\left(\frac{\partial^2\psi}{\partial x^2}\right)^2 + 4\mu^2QR\,\frac{\partial^2\psi}{\partial x^2} - \mu^2\Sigma^2\left(\frac{\partial\psi}{\partial x}\right)^2 - 4\Sigma^2QR\psi + 4Q^2R^2 = 0, \tag{4}$$

is a real TE for contributons. This claim is disputed by Williams[5] on the basis that "a transport equation implies certain physical characteristics" such as the physical interpretability of each term in the equation. In the case of neutral particle transport, these requirements of interpretability and solvability are mutually compatible, since the linear Boltzmann TE is solvable without resort to auxiliary equations, as well as interpretable in terms of the detailed particle transport processes. However, in the case of contributon transport, the controversy has arisen because, to date, no single contributon TE satisfying both requirements has been published. To a great extent, this concern over which equation should be called a TE is indeed more "an argument over semantics than substance."[5] Therefore, let us define the semantics first before we identify additional equations.

There appears to be no disagreement in the literature concerning the terms "Boltzmann equation" or "Boltzmann transport equation." The most general form of the Boltzmann equation is a particle conservation equation in phase space obtained by Boltzmann[6] in connection with the kinetic theory of gases. In nuclear reactor and radiation transport theory, only the linearized form of the Boltzmann equation, which is also called the Boltzmann transport equation or simply "the transport equation," is commonly used for whatever neutral particles (neutrons, photons, etc.) are considered. For example, the time-independent neutron transport equation for the neutron flux distribution $\phi(r,\mathbf{\Omega},E)$ has the well-known form[7]

$$\mathbf{\Omega}\cdot\nabla\phi + \Sigma_T\phi = \iint \Sigma_s(r;\mathbf{\Omega}'E' \to \mathbf{\Omega},E)\phi(r,\mathbf{\Omega}',E')d\mathbf{\Omega}'dE' + Q(r,\mathbf{\Omega},E) \ , \tag{5}$$

where the notation of Ref. 7 is used. Of course, this equation is still a particle balance equation in the phase space $(r,\mathbf{\Omega},E)$ and therefore allows a physical interpretation for each term. In addition, if the particle source distribution Q and the system boundary conditions are given, Eq. (5) allows the computation of the neutron flux distribution ϕ at every phase-space point.

In search for a TE for contributons, three schools of thought have emerged, attaching different criteria to the definition of such an equation:

A. A TE for contributons is a single equation that allows the computation of the distribution of contributons at every

[1]M. L. WILLIAMS and W. W. ENGLE, Jr., *Nucl. Sci. Eng.*, **62**, 92 (1977).

[2]S. A. W. GERSTL, "A New Concept for Deep-Penetration Transport Calculations and Two New Forms of the Neutron Transport Equation," LA-6628-MS, Los Alamos Scientific Laboratory (1976).

[3]A. B. CHILTON, *Nucl. Sci. Eng.*, **63**, 219 (1977).

[4]S. A. W. GERSTL, *Nucl. Sci. Eng.*, **64**, 798 (1977).

[5]M. L. WILLIAMS, *Nucl. Sci. Eng.*, **64**, 799 (1977).

[6]L. BOLTZMANN, *Vorlesungen über Gastheorie*, J. A. BARTH, Ed., Leipzig, Germany (1910).

[7]G. I. BELL and S. GLASSTONE, *Nuclear Reactor Theory*, Van Nostrand Reinhold Company, New York (1970).

phase-space point without requiring the solution of auxiliary equations. There is no requirement for physical interpretability of the terms in this equation, but such interpretability is not ruled out.

B. A TE for contributons is a single equation for some appropriate distribution function that describes the physical behavior of contributons and their transport processes and allows an interpretation of the physical significance for each term. There is no requirement for solvability of this equation without the use of auxiliary equations, but such solvability is not ruled out.

C. A TE for contributons may consist of a set of equations that allows the computation of the phase-space distribution of contributons such that each equation separately allows a physical interpretation relevant to the contributon transport process. Note that the set can include only a single equation.

This classification of types of transport equations might be viewed as three intersecting sets, or classes, as shown in Fig. 1, since certain attributes such as interpretability or solvability of the equations are nonexclusive. An equation that could be considered as belonging to the intersection of all three classes might be termed the purest form of a TE. The Boltzmann equation, Eq. (5), is such an equation if neutral particle transport is considered.

In the framework of the above classification, Eq. (1) might indeed be called a TE for contributons under definition B. However, Eq. (1) does not describe *all* relevant physical processes that govern contributon transport; it is merely a spatial continuity condition, as Williams and Engle[1] also concede. To integrate the description of all physical processes relevant to contributon transport, a set of four equations consisting of Eqs. (1) and (3) together with the forward and adjoint Boltzmann equations for ϕ and ϕ^+ has to be considered. It is then clear that this specific definition of a TE for contributons fits only into class C above. In contrast, Eq. (4) is consistent with class A of a TE for contributons.

Our continued research effort has produced several additional TEs for contributons that can also be described within the above classification. The set of equations (for monoenergetic contributon transport in slab geometry),

$$\mu \frac{\partial \phi}{\partial x} + \sigma\phi = S \ , \tag{6}$$

$$-\mu \frac{\partial \phi^+}{\partial x} + \sigma\phi^+ = S^+ \ , \tag{7}$$

and

$$\mu \frac{\partial \psi}{\partial x} = S\phi^+ - S^+\phi \tag{8}$$

or

$$\psi = \phi\phi^+ \ , \tag{9}$$

where S and S^+ are the forward and adjoint source terms including both external and scattering sources, obviously falls into category C. The set consisting of Eqs. (6), (7), and (9) is one that is effectively solved in Ref. 1.

It is also possible to write a Boltzmann-like TE for ψ in which each of the terms has physical significance.[8] This equation can be written as

$$\begin{aligned}&\mathbf{\Omega}\cdot\nabla\psi(r,\mathbf{\Omega},E) + \bar{\sigma}_T(r,\mathbf{\Omega},E)\psi(r,\mathbf{\Omega},E)\\ &\quad = \iint \bar{\sigma}_s(r;\mathbf{\Omega}'E' \rightarrow \mathbf{\Omega},E)\psi(r,\mathbf{\Omega}',E')d\mathbf{\Omega}'dE'\\ &\qquad + Q(r,\mathbf{\Omega},E)\phi^+(r,\mathbf{\Omega},E) \ ,\end{aligned} \tag{10}$$

where

$$\bar{\sigma}_s(r;\mathbf{\Omega}',E' \rightarrow \mathbf{\Omega},E) = \sigma_s(r;\mathbf{\Omega}',E' \rightarrow \mathbf{\Omega},E)\frac{\phi^+(r,\mathbf{\Omega},E)}{\phi^+(r,\mathbf{\Omega}',E')} \ , \tag{11}$$

and

$$\begin{aligned}\bar{\sigma}_T(r,\mathbf{\Omega},E) &= \bar{\sigma}_s(r,\mathbf{\Omega},E) + \bar{\sigma}_a(r,\mathbf{\Omega},E)\\ &= \iint \bar{\sigma}_s(r;\mathbf{\Omega},E \rightarrow \mathbf{\Omega}',E')d\mathbf{\Omega}'dE' + \frac{R(r,\mathbf{\Omega},E)}{\phi^+(r,\mathbf{\Omega},E)} \ .\end{aligned} \tag{12}$$

The term-by-term physical interpretation of Eq. (10) is obvious when compared to the Boltzmann equation [Eq. (5)]. Thus, Eq. (10) describes the physical behavior of contributons and their transport processes quantitatively, but cannot be solved without also solving the adjoint Boltzmann equation as an auxiliary equation. Consequently, this equation must fall into class B. It should be noted that Eq. (1) is just the continuity equation corresponding to the contributon transport equation, Eq. (10).

Finally, it is possible to write a single third-order equation for ψ through elimination processes from Eqs. (6) through (9) (Ref. 9). For the monoenergetic problem in slab geometry, this equation reduces to

$$\begin{aligned}&-\mu^3\frac{\partial}{\partial x}\left[\frac{1}{SS^+(a+b)}\frac{\partial^2\psi}{\partial x^2}\right] + \frac{1}{2SS^+}\\ &\quad \times\left\{\mu^2\frac{\partial}{\partial x}\left[\frac{1}{SS^+(a+b)}\frac{\partial(SS^+)}{\partial x}\right] + \frac{2ab}{a+b}\right\}\mu\frac{\partial\psi}{\partial x}\\ &\quad = -\frac{SS^+}{(a+b)^3}\mu\frac{\partial}{\partial x}\left[\frac{(a+b)^2}{SS^+}\right] \ ,\end{aligned} \tag{13}$$

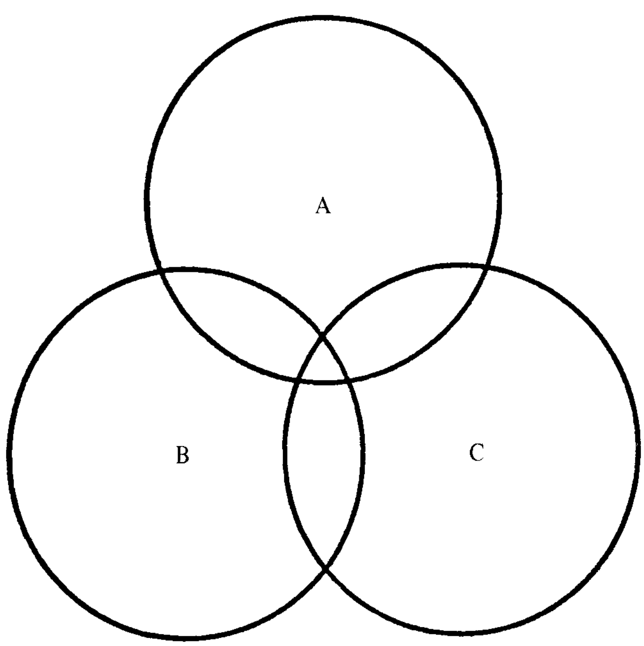

Fig. 1. A classification scheme for different types of TEs.

[8]W. F. MILLER, Jr. et al., "Transport and Reactor Theory October 1-December 31, 1977," LA-7131-PR, Los Alamos Scientific Laboratory (1978).

[9]W. F. MILLER, Jr. et al., "Transport and Reactor Theory April 1-June 30, 1978,", LA-7435-PR, Los Alamos Scientific Laboratory (1978).

for S and $S^+ \neq 0$, where

$$a = \frac{1}{S}\left(\mu \frac{\partial S}{\partial x} + \sigma S\right) \tag{14}$$

and

$$b = \frac{1}{S^+}\left(-\mu \frac{\partial S^+}{\partial x} + \sigma S^+\right) \ . \tag{15}$$

Equation (13) clearly falls into class A as a TE for contributons if S and S^+ contain only external sources. If, however, scattering sources are also considered, class C for this equation is more appropriate, because auxiliary equations need to be solved to determine the scattering source terms. Certain groups of terms in Eq. (13) also allow physical interpretation.

Other TEs for ψ can be obtained, depending on how the nonlinearity of contributon transport is handled. Also, different combinations of the basic equations, Eqs. (6) through (9), may obviously be manipulated such that other class C sets of equations are obtained.[10] Similarly, if the term "equation" is taken to be synonymous with "computational procedure," then certain Monte Carlo procedures[11] can also be considered as a TE for contributons. The "purest form" of a TE for contributons that might fit all three classes—A, B, and C—i.e., belonging to the intersection A ∩ B ∩ C in Fig. 1, has yet to be identified, to our knowledge.

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[10]W. F. MILLER, Jr. et al., "Transport and Reactor Theory January 1-March 31, 1978," LA-7271-PR, Los Alamos Scientific Laboratory (1978).

[11]A. DUBI, S. A. W. GERSTL, and D. J. DUDZIAK, *Nucl. Sci. Eng.*, **68**, 19 (1978).