could stop here. However, it is a simple matter of multiplication and addition to include up to $0(E^8)$ terms in this second iteration and improve the accuracy even further.

We thus obtain the expression for E_c to $0(E^8)$:

$$E_c = E_c\Big|_{[\text{from Eq. (4)}]} - \frac{C_3^2\, G_3 D_3}{1024}\left(\frac{3}{2} + \Delta\right)E^5 + \frac{G_3 C_3^2}{8192}\left[-\frac{C_3 G_3}{2}(1+\Delta) + D_3^2\right]E^6 + \frac{G_3^2 D_3 C_3^3}{8(8192)}E^7 + \frac{G_3^3 C_3^4}{8(262,144)}E^8 \quad . \qquad (8)$$

A similar expression is obtained for the right-hand side of Eq. (3) up to $0(E^7)$. The improved accuracies computed by including these higher-order terms in the second iteration are shown in column 6 of Table I. In column 7 we show the accuracy if one includes the sin $5\pi x$ correction

$$E_c = E_c\Big|_{[\text{from Eq. (8)}]} - \frac{C_5[1 + (\Delta/3)]}{24}E^2 \quad . \qquad (9)$$

Thus, reasonable accuracy can be obtained in the second iteration for this problem.

Louis M. Shotkin

Brookhaven National Laboratory
Upton, New York 11973

August 13, 1969

Comments on Theoretical and Experimental Criteria for Reactor Stability

Kalinowski[1] recently criticized the proof of a stability criterion given by Gyftopoulos.[2] But neither the arguments given by Kalinowski nor the reply by Gyftopoulos are satisfying because the solutions of the corresponding kinetic equations [Eqs. (1) through (3) in Ref. 2] are interpreted in a finite dimensional Euclidean state space. Since these equations represent a system of functional-differential equations, it is necessary to interprete the solutions in an appropriate function space [in this case $C(-\infty, 0]$].

Indeed Eqs. (1) through (3) of Ref. 2 are autonomous as Gyftopoulos says. This is due to the fact that (cf., Ref. 3, p. 764)

$$I = \int_{-\infty}^{t} f(t-\tau)p(\tau)d\tau = \int_{-\infty}^{0} f(-\tau)p(t+\tau)d\tau \quad .$$

$p(t+\tau)$, $-\infty < \tau \leq 0$, is a function in $(-\infty, 0]$ usually denoted by the symbol p_t. For any value of t the function p_t belongs to the space $C(-\infty, 0]$. Therefore, the integral I can be written as $I = F(p_t)$. If t varies, then I changes its value only if p_t varies as an element of $C(-\infty, 0]$.

The Liapunov functional V used by Gyftopoulos [Eq. (17) in Ref. 2], contrary to the statement of Kalinowski, is positive definite without any assumption over the integrals, if the given conditions on the parameters are fulfilled. If the calculations relating to the step from Eq. (19) to Eq. (20) of Ref. 2 were correct, then the time derivative of V would be only negative semidefinite and not, as Gyftopoulos says, negative definite. In fact, $\dot{V}$ is zero for $p(t) = 0$ and $c_i(t)$ $(i = 1, \ldots, m)$ arbitrarily. But apart from the correctness of Eq. (20), which will be discussed below, the methodological foundation of the paper by Gyftopoulos is wrong, because it is not possible to apply classical theorems of Liapunov's direct method to Eqs. (1) through (3) of Ref. 2, since these are functional-differential equations and not ordinary differential equations. In Ref. 3, considering Eq. (20) as correct, it is shown that Gyftopoulos' criterion can be proved applying an extension of Liapunov's direct method to functional-differential equations given by Hale (Ref. 4). Moreover, since Gyftopoulos provides no proof for the domain of asymptotic stability given in Ref. 2 [in fact the domain defined by the inequalities (23) is merely a domain where V is positive definite], in Ref. 3 there is given a domain which is surely contained in the domain of attraction relating to the power equilibrium state.

Some correspondence following up the publication of the paper quoted in Ref. 3 revealed an error in the step from Eqs. (A7) to (A9) in Ref. 2. Precisely, one has that from

$$|K(\omega_1,\omega_2)|^2 = 4\text{Re}\, G(j\omega_1)\text{Re}\, G(j\omega_2) + C^2(\omega_1,\omega_2) \qquad (1)$$

does not follow

$$K(\omega_1,\omega_2) = 2[\text{Re}\, G(j\omega_1)\text{Re}\, g(j\omega_2)]^{1/2} + jC(\omega_1,\omega_2) \quad , \qquad (2)$$

but more generally

$$(\omega_1,\omega_2) = 2[\text{Re}\, G(j\omega_1)\text{Re}\, G(j\omega_2)]^{1/2}\alpha(\omega_1,\omega_2) + C(\omega_1,\omega_2)\beta(\omega_1,\omega_2) \quad . \qquad (3)$$

$\alpha(\omega_1,\omega_2)$ and $\beta(\omega_1,\omega_2)$ are complex numbers with $|\alpha| = |\beta| = 1$. In other words, knowning $|K(\omega_1,\omega_2)|^2$ one can not determine uniquely $K(\omega_1,\omega_2)$.

The incorrectness of Eq. (20) in Ref. 2 is confirmed by the discussion of a particular case. For instance Gyftopoulos considers the case where b is very large. In fact, considering, as Gyftopoulos says, only the last integral term in V we have

$$V = \frac{b\lambda}{a}\int_{-\infty}^{t}\frac{a-1-p(\tau)}{1+p(\tau)}k^2(\tau)d\tau \qquad (4)$$

and

$$\dot{V} = \frac{b\lambda}{a} \times \frac{a-1-p(t)}{1+p(t)}k^2(t) \quad . \qquad (5)$$

Now, if we have $d^2 = a$ and $a, d > 1$, then

$$-1 < p(t) < d - 1 \qquad (6)$$

implies

$$-1 < p(t) < a - 1 \quad . \qquad (7)$$

Equations (5) and (7) prove that $\dot{V}$ is positive semidefinite, because b, a, λ are positive constants. But according to Eq. (20) of Ref. 2, $\dot{V}$ should be negative semidefinite!

It is interesting to mention that in the special case $b = 0$ the condition $\text{Re}\, G(\omega) > 0$ given by Gyftopoulos coincides with the condition relating to the new stability criterion given by the authors of this letter in Ref. 5, considering a new Liapunov functional and applying an extension of the stability theory given by Hale in Ref. 4. However the coincidence of $G(\omega) > 0$ with the new criterion in the

[1]JOSEPH E. KALINOWSKI, *Nucl. Sci. Eng.*, **34**, 200 (1968).

[2]E. P. GYFTOPOULOS, *Nucl. Sci. Eng.*, **26**, 26 (1966).

[3]F. DI PASQUANTONIO and F. KAPPEL, *Energia Nucleare*, **15**, 761 (1960).

[4]J. K. HALE, *J. Diff. Eqs.*, **1**, 452 (1965).

[5]F. DI PASQUANTONIO and F. KAPPEL, to be published in *Energia Nucleare*.

special case $b = 0$ must be interpreted as a casual coincidence. In fact in Ref. 5 it is shown that the Appendix given by Gyftopoulos in Ref. 2 is wrong also for $b = 0$.

It is important to note that in Ref. 3 there is provided a rigorous proof for the asymptotic stability criterion first given by Akcasu and Dalfes (Ref. 6) in 1960 and later used by Akcasu and Akhtar (Ref. 7) and Lellouche (Ref. 8) to investigate the stability of a xenon controlled point reactor with the presence of temperature feedback. In Ref. 3 it is pointed out that the criterion is correct but the proof given in Ref. 6 is incorrect, because the asymptotic stability is based on the conclusion that $dU(x)/dx \to 0$ as $x \to \infty$ if $U(x)$ is non-increasing and bounded below. This conclusion is not correct, as one can easily find counter examples. However, if $dU(x)/dx$ is uniformly continuous then $dU(x)/dx \to 0$ as $x \to \infty$ by a lemma given by Barbalat (Ref. 9). Since the uniform continuity of $dU(x)/dx$ was not proved or mentioned in Ref. 6 the criticism by Di Pasquantonio and Kappel was justified. In fact, Akcasu and Akhtar (Ref. 10) had already presented a rigorous derivation of the new criterion given by Akcasu and Dalfes following the same analytical method given in Ref. 6 and using Barbalat's lemma. Unfortunately the authors of Ref. 3 did not know of this proof, and hence, could not take it into account in their paper.

Finally the authors of this letter wish to thank Mr. Gyftopoulos and Mr. Akcasu for an interesting correspondence which led to a complete agreement about the matter treated in this letter.

F. Di Pasquantonio

ENEL-Direzione Studi e Ricerche
Centro Ricerca di Automatica
Via Beruto 18
I 20131 Milano (Italy)

F. Kappel

2. Lehrkanzel fur Mathematik
Technische Hochschule in Graz
Kopernikusgasse 24
A 8010 GRAZ (Austria)

July 16, 1969

[6]A. Z. AKCASU and A. DALFES, *Nucl. Sci. Eng.*, **8**, 89 (1960).

[7]A. Z. AKCASU and P. AKHTAR, *J. Nucl. Energy*, **21**, 341 (1967).

[8]G. S. LELLOUCHE, *J. Nucl. Energy*, **21**, 519 (1967).

[9]I. BARBALAT, *Rev. Math. Pures Appl.*, **4**, 267 (1959).

[10]A. Z. AKCASU and P. AKHTAR, *Proc. Brookhaven Conference on Industrial Needs and Academic Research in Reactor Kinetics*, April (1968) 8-9, BNL 50117 (T-497), to be published.

Comments on the Neutron Transport Theory with Anisotropic Scattering

In this letter, which is concerned with one-group neutron transport with anisotropic scattering in a homogeneous medium, it will be shown that there are no discrete eigenvalues of the transformed transport equation within the region of continuous eigenvalues. The existence of discrete eigenvalues in this region has been assumed by Mika,[1] who applied the spherical harmonics method to the aforementioned problem. Zelazny[2] has already shown that in the case of one-group neutron transport with isotropic scattering in a homogeneous medium, such eigenvalues do not exist.

Analogously to Mika,[1] the scattering function is expanded into a finite series of Legendre polynomials

$$f(\Omega' \to \Omega) = (4\pi)^{-1} \sum_{K=0}^{N} b_K p_K(\Omega' \times \Omega)$$

and introduced into Boltzmann's equation for plane geometry

$$\mu \times (\partial/\partial x)\, \Psi(x,\mu) + \Psi(x,\mu) = (c/2) \sum_{K=0}^{N} b_K p_K(\mu) \int_{-1}^{+1} p_K(\mu')\, \Psi(x,\mu')\, d\mu' \quad .$$

With help of the relation

$$\Psi(x,\mu) = \exp(-x/\nu)\, \phi(\nu,\mu) \quad ,$$

the transformed equation

$$(\nu - \mu) \times \phi(\nu,\mu) = (c\nu/2) \sum_{K=0}^{N} b_K p_K(\mu) \int_{-1}^{+1} p_K(\mu')\, \phi(\nu,\mu')\, d\mu' \tag{1}$$

is obtained. Multiplication with $p_s(\mu)$ and integration over μ from -1 to +1 yields

$$(s+1) h_{(s+1)}(\nu) - \nu[cb_s - 2(s+1)]\, h_s(\nu) + s h_{(s-1)}(\nu) = 0 \quad , \quad s = 0,1,2\ldots \tag{2}$$

where

$$h_K(\nu) = \int_{-1}^{+1} p_K(\mu)\, \phi(\nu,\mu)\, d\mu \quad , \quad K = 0,1,2\ldots \tag{3}$$

$$h_{-1}(\nu) = 0 \quad . \tag{4}$$

From Eqs. (2) to (4) $h_K(\nu) \sim h_0(\nu)$. Therefore, the normalization

$$h_0(\nu) = 1 \tag{5}$$

may be used without lose of generality. The general solution of (1) is

$$\phi(\nu,\mu) = (c/2)\, [\nu/(\nu - \mu)] \sum_{K=0}^{N} b_K p_K(\mu)\, h_K(\nu) + \lambda(\nu)\, \delta(\nu - \mu) \quad .$$

There are discrete eigenvalues ν_i outside the interval $[-1, +1]$ given by

$$\Omega(\nu_i) = 0 \quad \nu_i \neq [-1, +1] \quad , \tag{6}$$

where

$$\Omega(z) = 1 - cz \sum_{K=0}^{N} b_K Q_K(z)\, h_K(z) \quad . \tag{7}$$

Additionally, there is a continuum of eigenvalues ν in the region $[-1, +1]$ with

$$\lambda(\nu) = P\Omega(\nu) \quad \nu \epsilon [-1, +1] \quad . \tag{8}$$

The completeness of the resulting set of eigenfunctions of Eq. (1) has been shown by Mika.[1]

The nonexistence of discrete eigenvalues within the continuum of eigenvalues will be demonstrated by showing that the assumption

$$\lim_{Z \to \nu} \Omega(z) = 0 \quad \nu \epsilon [-1, +1]$$

for arbitrary N leads to a contradiction to the normalization Eq. (5). Application of Plemelj's formula to Eq. (7) yields

[1]J. R. MIKA, *Nucl. Sci. Eng.*, **11**, 415 (1961).

[2]R. ZELAZNY, *Nukleonika*, **11**, 2 (1966).