Letters to the Editor

On the Physical Criteria for the Limiting Critical Frequency of Neutron Waves

In a recent paper, Ahmed et al.[1] compute the dispersion law and corresponding energy spectra of neutron waves in beryllium. Such a wave can be represented in the one-dimensional form

$$\Phi(v,x,t) = \exp(i\omega t)[A(\omega)\phi(v,\omega)\exp(-ik(\omega)x) + \Phi_c(v,x)] \ , \quad (1)$$

where Φ_c is a nonseparable function of space and energy often referred to as "the contribution from the continuum."[2] As has been discussed in Ref. 2, there exists a frequency, $\omega^* \sim (v\Sigma_T)_{\min}$, above which it is possible that $\phi(v,\omega)$, the energy spectrum of the wave at frequency ω, vanishes identically in ω. There thus exists a limiting frequency, ω_0^*, at and above which Φ consists of the continuum contribution only. It is to be emphasized that $\omega_0^* \geq \omega^*$.

A similar situation exists in the case of the initial value problem in moderators, often referred to as the die-away experiment.[3] Here, the neutron density at long times behaves like

$$N(\boldsymbol{r},v,t;B^2) = \{n(v,B^2)\exp[-\lambda(B^2)t] + N_c(v,t)\}\exp(i\boldsymbol{B}\cdot\boldsymbol{r}) \quad , \quad (2)$$

where $N_c(v,t)$ is a nonseparable function of space and energy. It is argued in Ref. 3 that there exists a limiting buckling, B^*, such that for $B > B^*$, $N = N_c\exp(i\boldsymbol{B}\cdot\boldsymbol{r})$.

Both B^* and ω_0^* are uniquely determined by the particular moderator in question. Numerical estimates of B^* for various moderators have been made by calculating numerically the function Re $n(v,B^2)$ for various values of B. The limiting value B^* is then found from the condition that there exists a $v = v^*$, $0 \leq v^* \leq \infty$, such that

$$\text{Re}\ n(v^*,B^{*2}) < 0 \quad . \quad (3)$$

The justification for this criterion is derived from the following argument. By a suitable choice of initial and boundary conditions, it is always possible, after sufficient time has elapsed, for

$$N(r,v,t;B^2) = \text{Re}\ n(v,B^2)\cos B_x x\cos B_y y\cos B_z z\exp(-\lambda t) \ ,$$

where the dimensions of the medium are such that the spatial factor is nonnegative. Since the neutron density, N, is nonnegative, Re $n(v,B^2) \geq 0$. If, for the point (v^*,B^*), then the calculated n no longer describes N.

In an effort to find ω_0^* for beryllium, Ahmed et al.[1] make what appears to be a reasonable extension of the criterion[3] to the wave domain. They assume that ω_0^* is such that, for some $0 \leq v^* \leq \infty$,

$$\text{Re}\ \phi(v^*,\omega_0^*) < 0 \quad . \quad (4)$$

Although apparently reasonable, I believe the criterion to be incorrect. Unlike the case of the initial value problem, it is never possible in the wave problem to represent the neutron density by the real part of the discrete contribution to Eq. (1). Indeed, if this were so, the neutron density would become periodically negative at any frequency! The correct representation of the neutron density, using the most favorable combination of boundary and source conditions, is

$$N(x,v,t;\omega) = A'(\omega)\ \text{Re}\ \phi(v,\omega)\cos[\omega t - \xi(\omega)x + \theta(\omega)] \times\exp[-\alpha(\omega)x] + \text{Re}\ \phi(v,0)\exp(-x/L) \ , \quad (5)$$

where ξ and $-\alpha$ are the real and imaginary parts of k, L is the usual diffusion length [equal to $1/\alpha(0)$] and the functions $A'(\omega)$ and $\theta(\omega)$ are nontrivial amplitude and phases that determine the relative distribution of neutrons between the dc and ac components of the total neutron density. These two functions are determined only in part by the nature of the source and hence are not completely arbitrary.

The physically correct condition, $N(x,v^*,t;w_0^*) \leq 0$, for the critical frequency may be put in the form

$$\text{Re}\ \phi(v,0) \leq A'(\omega^*)\exp[\alpha(0) - \alpha(\omega_0^*)]x\,|\text{Re}\ \phi(v^*,\omega_0^*)| \ , \quad (6)$$

where x must be greater than the minimum distance to achieve an asymptotic spectrum.

A more convenient estimate[3] of ω_0^* is obtained from the dispersion law itself. It can be shown[4] that, in the one-dimensional case, the equation of the continuum is

$$\alpha_c = \Sigma_T(\omega/\xi_c) \quad , \quad (7)$$

where ξ_c and α_c are the real and imaginary parts of any spatial continuum eigenvalue. It is clear that as long as $\alpha(\omega) < (\Sigma_T)_{\min}$, $\omega < \omega_0^*$. The converse, if $\alpha(\omega^*) = (\Sigma_T)_{\min}$ then $\omega^* = \omega_0^*$, is, of course, not necessarily true. This $(\Sigma_T)_{\min}$ condition for the existence of discrete waves was discussed in Ref. 2 and is a confusing condition since it is a sufficient condition for the existence of discrete waves but not necessary. Because of Eq. (7), one need only plot α vs ω/ξ as obtained from the computed (or measured) dispersion law and on the same graph, plot Σ_T vs $v = \omega/\xi$. The point at which the two curves intersect $(\alpha^*,\omega_0^*/\xi)$ will determine ω_0^*. The condition, Eq. (7), comes directly out of one-dimensional transport theory and can be generalized to include more dimensions, as well as a restricted set of finite media.[4]

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[3] N. CORNGOLD, *Nucl. Sci. Eng.*, **19**, 80 (1963).

[4] M. N. MOORE, "Eigenfunction Degeneracy for Neutron Waves" (in preparation).