causes $D(v)$ to fluctuate, the derivative of $v\Sigma$ fluctuates even more sharply, and the effect upon $R(t)$ is to make it appear to be composed of discrete, exponential modes. This is the effect we seek. It should account for the experimental results without recourse to a cut-off in velocity[3].

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On the Use of the Poincaré-Bertrand Formula in Neutron Transport Theory

In a recent letter, Jacobs and McInerney[1] have questioned some of the results obtained by the normal-mode method[2] in one-speed neutron transport theory. For instance, the version of the full-range closure relation (for isotropic scattering), which is implicit in some previously reported results[2,3],

$$\frac{\phi(L,\mu)\phi(L,\mu')}{M_+} + \frac{\phi(-L,\mu)\phi(-L,\mu')}{M_-} + \int_{-1}^{1} \frac{\phi(\nu,\mu)\phi(\nu,\mu')}{M(\nu)} d\nu = \frac{\delta(\mu - \mu')}{\mu}, \quad (1A)$$

is criticized. Instead, the right-hand side should read as[1]

$$\frac{\lambda^2(\mu)}{M(\mu)} \delta(\mu - \mu'), \quad (1B)$$

in Mika's notation[3]. This criticism also applies to a number of previously established results for Green's function and albedo problems, where integrals similar to that in Eq. (1) appear in the expressions for the angular density.

The difference between (1A) and (1B) lies in the interpretation of Cauchy principal-value integrals, if the integrand has two singularities that are allowed to merge. Such integrals are handled by the Poincaré-Bertrand formula[4],

$$\int d\nu \int d\mu' F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} = \int d\mu' \int d\nu F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} + \pi^2 F(\mu,\mu), \quad (2B)$$

with μ inside the interval over which both integrations are carried out.

This formula is not completely clear until we define what is meant by the integral over ν on the right-hand side when $\mu' \to \mu$. This is done by using the identity[4]

$$P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} \equiv \frac{1}{\mu - \mu'}\left[P\frac{1}{\nu - \mu} - P\frac{1}{\nu - \mu'}\right], \quad (3B)$$

with the agreement that the limit $\mu' \to \mu$ may be carried out only after integration over ν.

Other definitions of the limit of that integral can be proposed that lead to an infinity like $\delta(\mu - \mu')$. Since there is some freedom in the choice of the definition, we take the liberty to modify Eq. (3B) in such a way that the extra term from the Poincaré-Bertrand formula is incorporated here. That is, we define[5]

$$P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} \equiv \frac{1}{\mu - \mu'}\left[P\frac{1}{\nu - \mu} - P\frac{1}{\nu - \mu'}\right] + \pi^2\delta(\nu - \mu)\delta(\nu - \mu'), \quad (3A)$$

so that (2B) is replaced by

$$\int d\nu \int d\mu' F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} = \int d\mu' \int d\nu F(\nu,\mu') P\frac{1}{\nu - \mu} P\frac{1}{\nu - \mu'} . \quad (2A)$$

As in version B, each side of Eq. (3A) applies to the corresponding side of Eq. (2A). That is, the left-hand side of Eq. (3A) can be used only if the integration over μ or μ' comes first, whereas we use the right-hand side if the integration over ν is to be carried out first.

To summarize, we now have two versions of the Poincaré-Bertrand formula: Eqs. (2B) and (3B) or, alternatively, (2A) and (3A). With either version, a consistent system of formulas for neutron transport theory can be constructed. Jacobs and McInerney have demonstrated this for version B, and several earlier authors for version A. For example, in the two versions the integrand occurring in Eq. (1) is analyzed according to the following identities:

$$\phi(\nu,\mu)\phi(\nu,\mu') \equiv \frac{c\nu}{2}\frac{1}{\mu - \mu'}\left[\phi(\nu,\mu) - \phi(\nu,\mu')\right] + \left[\lambda^2(\mu) + \left(\frac{1}{2}\pi c\mu\right)^2\right]\delta(\nu - \mu)\delta(\nu - \mu'), \quad (4A)$$

$$\phi(\nu,\mu)\phi(\nu,\mu') \equiv \frac{c\nu}{2}\frac{1}{\mu - \mu'}\left[\phi(\nu,\mu) - \phi(\nu,\mu')\right] + \lambda^2(\mu)\delta(\nu - \mu)\delta(\nu - \mu'). \quad (4B)$$

This explains the difference between Eqs. (1A) and (1B).

For neutron transport theory, version A is to be recommended for two reasons. The first is tradition; except for the work of Jacobs and McInerney[1,6], version A has been used consistently in this field, although sometimes without due explanation. Secondly, many formulas and derivations are much simpler and shorter in this version because Eq. (2A) permits us to formally switch orders of integration.

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A Note on the Adjoint Function in the Time Optimal Xenon Shutdown Problem

Smith and Roberts[1] (hereinafter I) have recently applied the Pontryagin theorem to time optimal xenon shutdown in

[1] J. J. ROBERTS and H. P. SMITH, Jr., "Time Optimal Solution to the Reactivity-Xenon Shutdown Problem," *Nucl. Sci. Eng.*, **22**, 470 (1965).

an elegant and successful way. The purpose of this note is to point out one aspect of the adjoint functions used in the analysis in respect to their reversed-time problem.

From Eqs. (2) and (11) of I we have Hamiltonian densities in forward ($t+$) and backward time ($t-$) of the form

$$H(t+) = \mathbf{p}(t+)\cdot\mathbf{f}(t+);\ H(t-) = -\,\mathbf{p}(t-)\cdot\mathbf{f}(t-). \tag{1}$$

But both are to satisfy Theorem 1, that $H \geqslant 0$. Clearly, therefore, the adjoint functions are not the same in the two times but rather $p(t+) = -p(t-)$, $f(t)$ having even parity. Therefore when Smith and Roberts determine the sign of p_2, it is rather significant to ensure that they are treating the correct p_2. This may be done most easily perhaps by evaluating the Hamiltonian densities at the final time, t_1, when the shutdown trajectory meets the target curve Ω.

It is clear from Fig. 1 of I that if the target curve is intersected by the final trajectory in the region of interest then the slopes must satisfy

$$\left.\frac{dX}{dI}\right|_{\phi} \leqslant \left.\frac{dX}{dI}\right|_{\Omega} \text{ at } t_1, \tag{2}$$

where subscript ϕ indicates the shutdown at maximum flux. But the boundary condition on the adjoint functions obtained from the transversality conditions (the same in forward and reversed time) is

$$p_2 \left.\frac{dX}{dI}\right|_{\Omega} + p_1 = 0 \text{ at } t_1. \tag{3}$$

Furthermore, from Fig. 4 of I, dI/dt is positive at t_1 in the region of interest.

Evaluating $H(t-)$ we have

$$\begin{aligned} H(t-) &= -\,p_1(t-)\frac{dI}{dt} - p_2(t-)\frac{dX}{dt} \text{ at } t_1 \\ &= \frac{dI}{dt}\,p_2(t-)\left[\left.\frac{dX}{dI}\right|_{\Omega} - \left.\frac{dX}{dI}\right|_{\phi}\right]. \end{aligned} \tag{4}$$

Then for a positive H we must generally take $p_2(t_1-)$ to be positive, confirming the supposition of I. Furthermore, if $H(t_1+)$ is to be positive, we must take $p_2(t_1+)$ to be negative as already predicted.

Note that the sign is not determined by the transversality condition alone since orthogonality is equally satisfied by the inward as the opposite outward pointing vector.

The resulting adjoint functions in real time are therefore negative through the control period, conflicting with our usual ideas of perturbation theory and the importance of a source of iodine or xenon. This difficulty can be removed on making an obvious change in the optimization theorem. We now require that for a minimum control period, the Hamiltonian is to take its least value as a function of the flux and that the result is not positive:

$$H = \inf_{\phi\epsilon\Phi} H(\mathbf{p}, \mathbf{x}, \phi) \leqslant 0. \tag{5}$$

Then in real time the adjoint functions will be positive (negative in reverse time). This restatement leaves the switching points unaffected, of course, but is somewhat more in line with the 'traditional' calculus of variations for isoperimetric problems.

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