loses its power. We have chosen $G = 25$ as a compromise between these considerations.

Many of the Monte Carlo estimators that arise in our work have probability distributions about which little is known. It seems wise, therefore, to use a test that is powerful against a wide variety of non-normal distributions. Shapiro and Wilk have studied the power of their test against a wide variety of alternative distributions that arise in statistical theory (e.g., log normal, chi square with 1, 2, 4, and 10 degrees of freedom, noncentral chi square, Cauchy, exponential, rectangular, Poisson, and others). They show the test to be powerful, although of course not equally powerful, against all of them[1]. Our confidence in the test is fundamentally based on their study.

Our experience has added to this confidence. We have been applying the test to Monte Carlo results for over a year. We have not yet had any cases in which a confidence limit that had not been flagged by the normality test turned out to be bad.

For the convenience of those who have not immediate access to the reference, the test of Shapiro and Wilk for a sample of size 25 is:

Let $z_1, z_2, \ldots, z_{25}$ be the 1st, 2nd, . . . , 25th estimators. Arrange the z's in increasing order of magnitude; relabel as $y_1, y_2, \ldots, y_{25}$ where $y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_{25}$. Compute

$$b = \sum_{i=1}^{25} a_i y_i$$

and

$$S^2 = \sum y_i^2 - \frac{\left(\sum y\right)^2}{25}$$

where the $a_i = -a_{26-i}$ are given below.

i	a_i
1	-0.4450
2	-0.3069
3	-0.2543
4	-0.2148
5	-0.1822
6	-0.1539
7	-0.1283
8	-0.1046
9	-0.0823
10	-0.0610
11	-0.0403
12	-0.0200
13	0.0000

The test for departure from normality is based on the statistic

$$w_{25} = b^2/S^2.$$

The 1%, 5% and 10% significance values of w_{25} are 0.888, 0.918 and 0.931, respectively (values smaller than critical are significant).

Our experience with this test has been good, and we recommend the use of this test when confidence intervals are computed by the approach referred to here.

G. L. Burrows
D. B. MacMillan

Knolls Atomic Power Laboratory*
General Electric Company
Schenectady, New York

Received December 3, 1964
Revised February 4, 1965

*Operated for the USAEC by the General Electric Company.

On the Inclusion of Boundary Terms in Time-Dependent Synthesis Techniques

In a recent paper[1] dealing with the application of synthesis techniques to various time-dependent problems, Kaplan, Marlowe, and Bewick presented a variational principle for linear time-dependent group-diffusion theory. The principle, however, is not stationary with respect to arbitrary variations in the functions involved because not all of the end-point (in time) terms in the first variation vanish. The authors postulate that this difficulty can be removed if the variations are limited to functions having the same end points as the selected function. Implicit in such an argument, however, is the assertion that the trial functions which are to be assumed will lead to an approximate solution having the same end-point values as has the exact solution. This is a requirement not easily met.

The difficulty can be avoided by the inclusion of appropriate boundary terms in the functional. For simplicity, consider the one-group-flux and adjoint-flux equations without delayed neutrons:

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = \nabla \cdot D \nabla \phi + (\nu F - A)\phi \qquad (1)$$

$$-\frac{1}{v}\frac{\partial \phi^*}{\partial t} = \nabla \cdot D \nabla \phi^* + (\nu F - A)^* \phi^* , \qquad (2)$$

[1]S. KAPLAN, O. J. MARLOWE and J. BEWICK, *Nucl. Sci. Eng.*, 18, 163-176 (1964).

where F and A are production and destruction operators. The boundary conditions are that ϕ and ϕ^* both vanish on the outer boundary S, and that the initial value of ϕ and the final value of ϕ^* are to be specified.

We now invoke the functional

$$J = \int_{t_a}^{t_b} dt \int_R d\boldsymbol{r}\Big[- (\nabla\phi^*)\cdot D\nabla\phi + \phi^*(\upsilon F - A)\phi - \frac{1}{2v}\left(\phi^* \frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\Big] + \frac{1}{2v}\int_R d\boldsymbol{r}[\phi^*(\boldsymbol{r},t_b)\phi(\boldsymbol{r},t_b) + \phi^*(\boldsymbol{r},t_a)\phi(\boldsymbol{r},t_a)]. \tag{3}$$

The functional used in Ref. 1 does not include the second integral (the boundary terms) on the right side of Eq. (3). The variations with respect to ϕ^* and ϕ yield the Euler equations (1) and (2), the spatial boundary conditions

$$\int_{t_a}^{t_b} dt \int_S dS\, \delta\phi^*\, D\nabla\phi = 0 \tag{4}$$

$$\int_{t_a}^{t_b} dt \int_S dS\, \delta\phi\, D\nabla\phi^* = 0 \tag{5}$$

and the temporal boundary conditions

$$\frac{1}{2v}\int_R d\boldsymbol{r}[\phi(\boldsymbol{r},t_b)\delta\phi^*(\boldsymbol{r},t_b) - \phi(\boldsymbol{r},t_a)\delta\phi^*(\boldsymbol{r},t_a)] + \frac{1}{2v}\int_R d\boldsymbol{r}[\phi(\boldsymbol{r},t_b)\delta\phi^*(\boldsymbol{r},t_b) + \phi(\boldsymbol{r},t_a)\delta\phi^*(\boldsymbol{r},t_a)] = 0, \tag{6}$$

$$-\frac{1}{2v}\int_R d\boldsymbol{r}[\phi^*(\boldsymbol{r},t_b)\delta\phi(\boldsymbol{r},t_b) - \phi^*(\boldsymbol{r},t_a)\delta\phi(\boldsymbol{r},t_a)] + \frac{1}{2v}\int_R d\boldsymbol{r}[\phi^*(\boldsymbol{r},t_b)\delta\phi(\boldsymbol{r},t_b) + \phi^*(\boldsymbol{r},t_a)\delta\phi(\boldsymbol{r},t_a)] = 0. \tag{7}$$

In Eqs. (6) and (7), the first integrals arise from integrating the $\frac{\phi}{v}\frac{\partial\delta\phi^*}{\partial t}$ and $\frac{\phi^*}{v}\frac{\partial\delta\phi}{\partial t}$ terms by parts, while the second integrals come from variation of the boundary terms. Since some of the terms cancel, we are left with

$$\frac{1}{v}\int_R d\boldsymbol{r}\, \phi(\boldsymbol{r},t_b)\delta\phi^*(\boldsymbol{r},t_b) = 0 \tag{8}$$

$$\frac{1}{v}\int_R d\boldsymbol{r}\, \phi^*(\boldsymbol{r},t_a)\delta\phi(\boldsymbol{r},t_a) = 0. \tag{9}$$

The functional Eq. (3) is thus stationary with respect to arbitrary variations in the functions ϕ and ϕ^* provided only that the admissible set of functions for ϕ and ϕ^* be restricted to those which satisfy the initial-value conditions imposed on ϕ and the final-value condition imposed on ϕ^* respectively, i.e., that the variations be taken such that

$$\delta\phi(\boldsymbol{r},t_a) = 0 \tag{10}$$

$$\delta\phi^*(\boldsymbol{r},t_b) = 0. \tag{11}$$

Since the adjoint problem is a final-value problem, one specifies final-value conditions in accordance with the interpretation one wishes to assign to the adjoint function[2]. Use of the functional Eq. (3) does not imply knowledge of any function at both end points of the time interval (t_a, t_b).

It is interesting to note that the formal procedure of Ref. 1 is not affected by the inclusion of the boundary terms in the functional Eq. (3). One makes the expansions

$$\phi(\boldsymbol{r},t) = \sum_{k=1}^{K} H_k(\boldsymbol{r})T_k(t) \tag{12}$$

$$\phi^*(\boldsymbol{r},t) = \sum_{k=1}^{K} H_k^*(\boldsymbol{r})T_k^*(t), \tag{13}$$

and one obtains the same set of equations for $T_k(t)$ and $T_k^*(t)$ as one would obtain using the principle of Ref. 1. It is thus seen that the results of Ref. 1 rest on a sounder theoretical base than was originally believed.

Martin Becker

General Electric Company
Knolls Atomic Power Laboratory*
Schenectady, New York

Received November 20, 1964

*Operated for the USAEC by the General Electric Company, Contract No. W-31-109 Eng-52.

[2]M. BECKER, *The Principles and Applications of Variational Methods*, Appendix A, MIT Press, Cambridge, Mass. (1964).

On Cadmium-Ratio Measurements for U^{235} and U^{233} Fission by Fission-Product Gamma Counting

In most determinations of fissile infinite dilution resonance integrals it is necessary to measure a cadmium ratio. One experimental method involves irradiating foils in the spectrum of interest, and subsequently gamma counting the