TABLE I

Magnitude of Auto-Spectral-Density Function for a Localized Absorber*

Frequency rad/sec	Space Point 4	Space Point 5
900	12.8	5.13
1000	8.13	2.94
2000	0.267	0.00124
3000	0.526	0.0231
4000	0.617	0.0256

*Tabulated value 10^3 times calculated value.

TABLE II

Real and Imaginary Parts of Cross-Spectral-Density Function*

Frequency rad/sec	Space Point 4		Space Point 5	
	Real	Imaginary	Real	Imaginary
200	1633	647.8	1081	225.8
300	460.7	348.2	333.7	135.5
400	123.9	202.7	115.1	89.60
500	11.16	125.2	39.07	63.20
600	-27.92	81.20	10.41	46.61
700	-40.09	54.71	-0.5946	35.46
800	-41.88	37.98	-4.567	27.56
900	-39.65	26.98	-5.647	21.73
1 000	-36.01	19.48	-5.549	17.31
2 000	-8.497	0.3338	-1.115	1.792
3 000	-0.7381	-0.8469	-0.3364	-0.2393
4 000	1.150	-0.6808	-0.1874	-0.4136
5 000	1.464	-0.4642	-0.1335	-0.3304
6 000	1.367	-0.3143	-0.1026	-0.2394
7 000	1.185	-0.2178	-0.08145	-0.1721
8 000	1.005	-0.1553	-0.06600	-0.1255
9 000	0.8514	-0.1139	-0.05438	-0.09341
10 000	0.7241	-0.08563	-0.04545	-0.07098

*Tabulated values 10^3 times calculated values.

600 rad/sec and again at 4000 rad/sec for space point 4 and at 700 rad/sec for space point 5. This frequency corresponds approximately to the sink frequency that has been observed for the coupled-core Argonaut reactor.[3,4] It would appear that the imaginary parts of the cross-spectral-density function may be particularly sensitive to the convergence of the modal solution since they are larger than might be expected for near symmetric locations.

The observation that the characteristics of the cross-spectral-density function are related to the degree of coupling of the fuel regions is certainly valid. An investigation of the effects of core spacing and the nuclear properties of the coupling region on the spectral functions is presently being performed.

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April 23, 1969

[3] R. A. HENDRICKSON and G. MURPHY, *Nucl. Sci. Eng.*, **31**, 215 (1968).

[4] R. W. ALBRECHT and W. SEIFRITZ, *Japan-U.S. Seminar on Nuclear Reactor Noise Analysis*, pp. 285,(1968).

On the Stabilizing Effect of Delayed Neutrons

For a nuclear reactor, the overall transfer function, which relates the Laplace transform of incremental power or neutron density $\delta n(s)$ to the Laplace transform of reactivity input $k(s)$, may be expressed as

$$\delta n(s)/[k(s)] = Z(s)/[1 + K(s)Z(s)] \ , \qquad (1)$$

provided that the block diagram shown in Fig. 1 represents the linear incremental model of the system. $Z(s)$, defined by

$$Z(s) = \frac{n_0}{l^*\left(s + \sum_i \frac{\beta_i s}{s + \lambda_i}\right)} \ ,$$

is the zero-power transfer function of the reactor, and $K(s)$ denotes the transfer function of the feedback block. Obviously, the operating power level or the neutron density of the reactor is indicated by n_0.

It is claimed by Smets[1] that delayed neutrons may exert a destabilizing effect upon a reactor system, if the system has an open-loop frequency characteristic which intersects the negative real axis twice in the form of curve A in Fig. 2a.

This conclusion was derived because curve A, which is the Nyquist plot of the system with the effects of delayed neutrons neglected, indicates a stable system, while curve B, obtained after the effects of delayed neutrons have been taken into account, reveals instability of the same power level.

It is true, at the power level n_0, the reactor without delayed neutrons is stable and the reactor with delayed neutrons is not. Such a result, however, is not sufficient to compare the degree of stability of the systems with the specified open-loop frequency characteristics, because, these are conditionally stable systems with two different stability regions.[2] Each system is stable for sufficiently low and sufficiently high power levels; that is, when both intersections of the Nyquist plot with the real axis are either to the right or to the left of the point $(-1 + j\ 0)$, it becomes unstable for a finite range of power between these two stability regions. Attention must be paid to the interesting fact that the range of power which corresponds to instability is different for each system. Therefore, the power level n_0 may lie in the instability region of the system with delayed neutrons, while it is within the stable power range of the system without delayed neutrons. At a different power level the results may reverse. For example, curves A′ and B′ in Fig. 2a indicate that at the power level $\frac{2}{3}\, n_0$, the system with delayed neutrons is stable, while the other system is not.

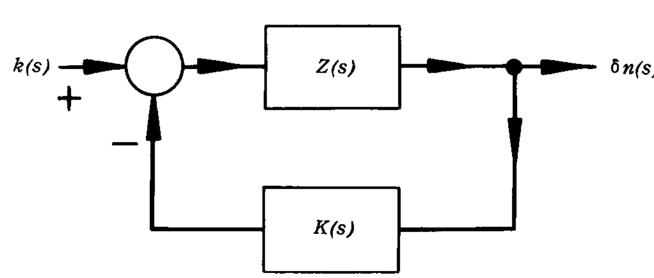

Fig. 1. Linear incremental model of a nuclear reactor.

[1] H. B. SMETS, *Nucl. Sci. Eng.*, **25**, 236 (1966).

[2] J. MILDA and N. SUDA, *Nucl. Sci. Eng.*, **11**, 55 (1960).

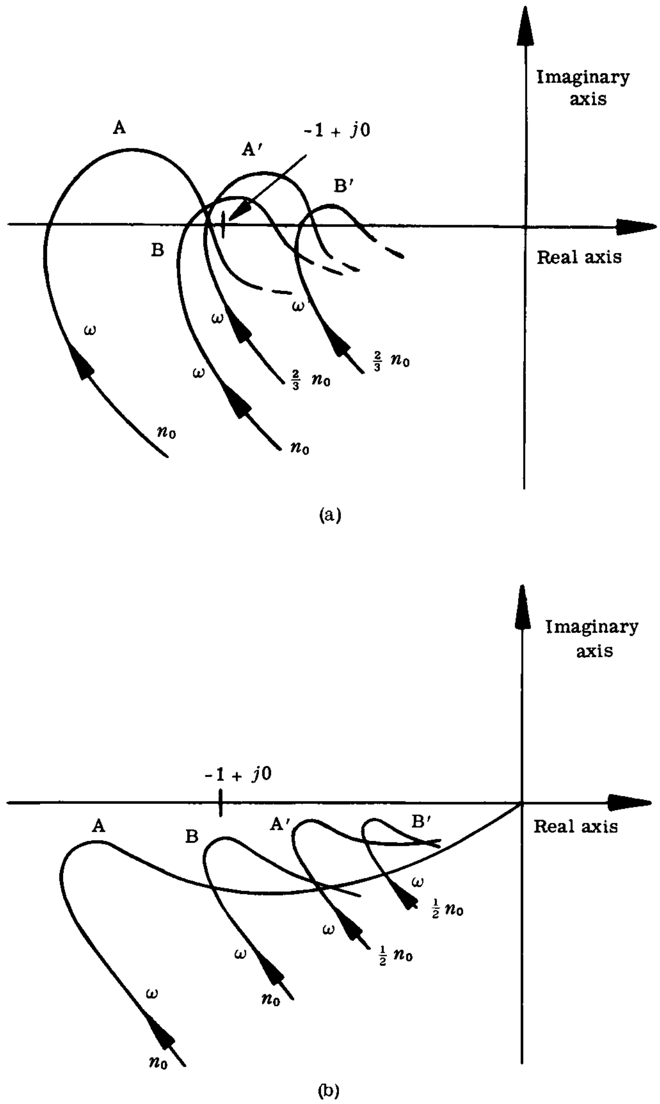

Fig. 2. Nyquist plots of a nuclear reactor neglecting and taking into account the effects of delayed neutrons. Curves A and A′ are the Nyquist plots without delayed neutrons. The Nyquist plots when delayed neutrons are taken into account are denoted by curves B and B′. n_0 is the reactor power level.

If power is gradually increased starting from zero, first the system without delayed neutrons becomes unstable. This shows that the critical power which corresponds to instability is larger under the effect of delayed neutrons. Besides, the system with delayed neutrons is unstable in a smaller power range compared with the system without delayed neutrons. If the static sensitivities of the systems are compared it is seen that because of the reduced open-loop gain the system with delayed neutrons is less sensitive. In view of all these facts, the system with delayed neutrons has to be considered more stable. Hence, such an example is not acceptable as an evidence of the destabilizing effects of delayed neutrons.

The curves in Fig. 2b represent the Nyquist plots for two reactors which are absolutely stable; that is, stable at all power levels. The only difference between these two systems is assumed to be the delayed neutrons. Curve A belongs to the system without delayed neutrons and curve B to the system with delayed neutrons. As seen, at a power level n_0 the system with delayed neutrons is less stable. If, because of this result, the conclusion is derived that the delayed neutrons reduce the degree of stability, this contradicts even Property 1, mentioned by Smets.[1]

These systems must also be compared when power is increased gradually, starting from zero. Under these conditions, it is seen that the degree of stability decreases first in the system without delayed neutrons. Curves A′ and B′ in Fig. 2b reveal this fact.

The Nichols chart was used by Smets[1] to investigate the effect of delayed neutrons upon the dynamic behavior of the linear incremental model shown in Fig. 1, for

$$K(s) = C\frac{(s + 0.0001)(s + 1)}{(s + 0.01)(s + 100)} ,$$

where C is a constant.

The conclusion derived, however, that with the type of feedback considered, delayed neutrons enhance the oscillatory behavior of the reactor, is not acceptable. The incorrect result is due to the incorrect determination of the arguments of the open-loop frequency characteristics.

Besides, it must also be mentioned that, even from a correctly determined resonant peak, it is hard to interpret the dynamic behavior of a system whose over-all transfer function is not simple. Only for a second-order feedback control system which has a closed-loop transfer function

$$\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2) ,$$

where ω_n is the undamped natural frequency, the resonant peak M_p and the resonant frequency ω_p are uniquely related to the damping ratio ζ.[3,4] The presence of additional poles and zeros will obviously alter the results. For instance, one system may have an M_p of 0.5 and a ζ of only 0.1, and another system may have an M_p of 1.5 and a ζ as great as 0.70.

Therefore, before resorting to rather tedious procedures to obtain the closed-loop frequency data, first, the open-loop frequency characteristic itself must be very carefully investigated, and as much information as possible as to the stability and the degree of stability of the system must be derived from it. If it is believed that the closed-loop frequency response also yields additional and significant information, it can be obtained later.

The following analysis shows that the effect of delayed neutrons upon the dynamic performance of the system can be very easily determined by interpreting the open-loop frequency data only.

Here, for the sake of simplicity, one mean group of delayed neutrons with parameters $\lambda = 0.08\ \mathrm{sec}^{-1}$ and $\beta = 0.00755$ will be considered. For $l^* = 10^{-3}$ sec, the open-loop transfer functions of the systems without and with delayed neutrons become

$$Z_1(s)\,K(s) = \frac{n_0 C}{l^*}\,\frac{(s + 0.0001)(s + 1)}{s(s + 0.01)(s + 100)}$$

and

$$Z_2(s)\,K(s) = \frac{n_0 C}{l^*}\,\frac{(s + 0.0001)(s + 1)(s + 0.08)}{s(s + 0.01)(s + 100)(s + 7.55)} ,$$

respectively.

The Bode plots of arguments of these open-loop transfer functions are shown in Fig. 3. Since there is no possibility

[3] O. J. M. SMITH, *Feedback Control Systems*, p. 26. McGraw-Hill Book Company Inc., New York, Toronto, London (1958).

[4] B. C. KUO, *Automatic Control Systems*, second ed., p. 392. Prentice-Hall Inc., Englewood Cliffs, N.J. (1967).

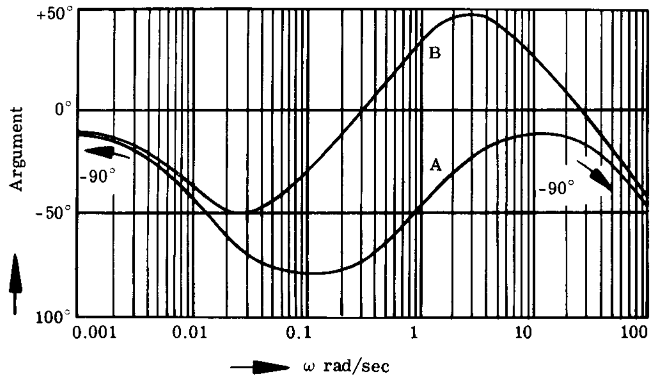

Fig. 3. Bode plots of arguments. Curves A and B represent the argument vs frequency diagrams for

$$\frac{1}{j\omega}\frac{(j\omega + 0.0001)(j\omega + 1)}{(j\omega + 0.01)(j\omega + 100)}$$

and

$$\frac{1}{j\omega}\frac{(j\omega + 0.0001)(j\omega + 1)(j\omega + 0.08)}{(j\omega + 0.01)(j\omega + 100)(j\omega + 7.55)},$$

respectively.

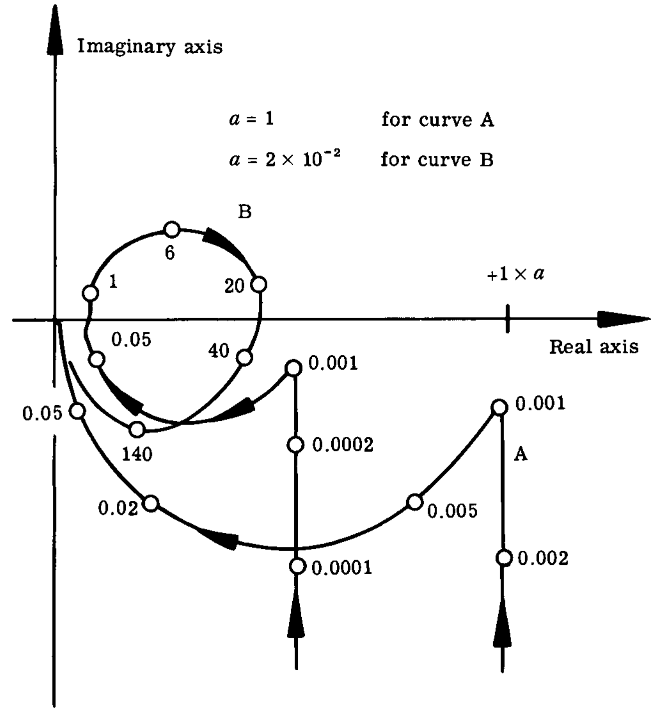

Fig. 4. Nyquist plots of highly overdamped systems. The diagrams for

$$\frac{(s + 0.0001)(s + 1)}{s(s + 0.01)(s + 100)}$$

and for

$$\frac{(s + 0.0001)(s + 1)(s + 0.08)}{s(s + 0.01)(s + 100)(s + 7.55)}$$

are denoted by curves A and B, respectively.

The figures on the curves are the values of frequency in radians per second.

for the arguments to be ±180°, it can be readily concluded that both of the systems are stable regardless of the value of the constant n_0C/l^*, that is, they are absolutely stable.

Again, the investigation of the arguments reveals that the Nyquist plots of the systems considered must remain in the right-hand half plane, and therefore, oscillatory transients can not be expected. In other words, these systems are both very overdamped. The Nyquist diagrams plotted for $n_0C/l^* = 1\ \text{sec}^{-1}$ in Fig. 4 are in good agreement with this conclusion.

Further, the open-loop transfer functions show that the reactor with delayed neutrons has a velocity gain constant which is smaller than the velocity gain constant of the system without delayed neutrons, by a factor of 0.08/7.55. Hence, the system with delayed neutrons is even more sluggish.

The peak in the closed-loop gain curve (Fig. 7, Ref. 1) obtained considering the effect of delayed neutrons is not the resonant peak, since the gain assumes larger values than this as the frequency is reduced. Therefore, it can not reveal an unfavorable effect of the delayed neutrons.

Again, if the feedback transfer function is

$$K(s) = C\frac{s + 1}{(s + 0.01)(s + 100)},$$

the delayed neutrons exert a stabilizing effect.

An alternative approach will be followed to analyze this example; namely, the Root-Locus method will be employed.

The open-loop transfer functions, in this case, without and with the delayed neutrons, respectively, are

$$Z_1(s)\,K(s) = \frac{n_0C}{l^*}\,\frac{(s + 1)}{s(s + 0.01)(s + 100)}$$

and

$$Z_2(s)\,K(s) = \frac{n_0C}{l^*}\,\frac{(s + 1)(s + 0.08)}{s(s + 0.01)(s + 100)(s + 7.55)}.$$

The Root-Locus diagram for the characteristic equation $1 + Z_1(s)K(s) = 0$ is sketched in Fig. 5a. Figure 5b denotes the same diagram for $1 + Z_2(s)K(s) = 0$.

As seen, the systems considered are stable for all values of the constant n_0C/l^*, in other words, the characteristic equation $1 + Z_1(s)K(s) = 0$ and $1 + Z_2(s)K(s) = 0$ can never have positive real roots or pairs of complex conjugate roots with positive real parts.

The value of the constant n_0C/l^*, which corresponds to critical damping, is indicated in Fig. 5 for each system. It is $0.0049\ \text{sec}^{-1}$ for the system without delayed neutrons and $0.49\ \text{sec}^{-1}$ for the system with delayed neutrons.

Figure 5 further reveals that the minimum decay constant of transient oscillations is $0.0049\ \text{sec}^{-1}$ in the case of the system without delayed neutrons. When delayed neutrons are taken into account the minimum decay constant of dominant transient oscillations becomes $0.0052\ \text{sec}^{-1}$.

The comparison of the power levels at which the systems become critically damped and of the decay constants show that the reactor with delayed neutrons is more stable than the other.

By comparing the closed-loop gain curves, Smets[1] has come to a reverse conclusion. However, if Fig. 8 in his paper is more carefully examined, it can be seen that the gain curve without delayed neutrons has a larger resonant peak—"distinct" or not it does not matter—than the other. If such an analysis were sufficient to estimate the effect of delayed neutrons upon the dynamic performance of the

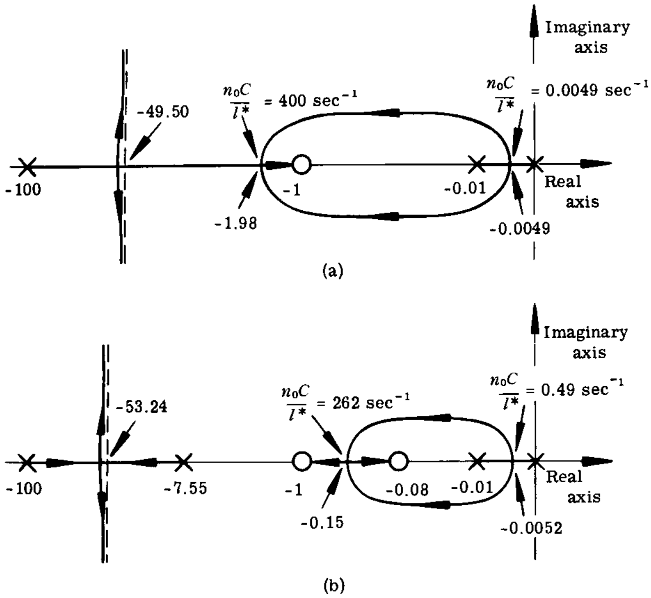

Fig. 5. Root-locus diagrams.

(a) A sketch of the root-loci for

$$1 + \frac{n_0C}{l^*} \frac{(s+1)}{s(s+0.01)(s+100)} = 0$$

(b) A sketch of the root-loci for

$$1 + \frac{n_0C}{l^*} \frac{(s+1)(s+0.08)}{s(s+0.01)(s+100)(s+7.55)} = 0 \ .$$

system, this would reveal the stabilizing rather than the destabilizing effect of delayed neutrons.

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April 21, 1969

"Reply to 'On the Stabilizing Effect of Delayed Neutrons' "

In the above Letter to the Editor, Professor Tan shows that the so-called "stabilizing effect of delayed neutrons" can be considered from vastly different angles, and she studies the variation of the *degree of stability when the power level is increased.* While this analysis is basically correct, it would be wrong to believe that it contradicts the conclusions given in Ref. 1, because in this reference the interest lay only in *comparing the dynamic properties* of reactors with and without delayed neutrons *at the same power level,* knowing that reactors without delayed neutrons did not exist and that, therefore, the comparison was fairly academic.

More precisely, in relation to Fig. 2a, above, Professor Tan agrees with the single conclusion regarding conditional stability reactors given in Ref. 1. The ensuing discussion by Professor Tan on the degree of stability simply completes the analysis given previously where this aspect was not considered. To conclude, as above, that this "example is not acceptable as an evidence of the destabilizing effects of delayed neutrons" seems, therefore, illogical because the reactor model does become unstable when the parameter β is increased from zero to its real, non-zero, physical value.

In Ref. 1, two examples of reactors in which delayed neutrons enhance the oscillatory behavior of the transient response are given. With the feedback kernel given erroneously in the caption of Fig. 6, it is true that the enhancement of the oscillations cannot be shown. However, it is easily seen that for reactors having feedbacks similar to the one drawn up in Fig. 6 of Ref. 1, the effect of delayed neutrons can be to make the response more oscillatory, (cf., curves A and B of Fig. 2b, above).

A second example is dealt with in Ref. 1, and it is concluded that "delayed neutrons enhance the oscillatory behavior of the solutions." To write that Smets has come to the conclusion that the reactor without delayed neutrons is more stable than with delayed neutrons is groundless, because it was not written and because the question of the degree of stability was not even considered.

This example can also be examined in terms of the root locus, provided that one does not assume, as done above, that ω is given in terms of rad/sec. The high frequency part of Figs. 7 and 8 (Ref. 1) shows that $\lambda > 100$ and, therefore, that the zero at $-\lambda$ and the pole as $-(\beta/l)$ of Fig. 5b above are at the left of the pole -100. Hence, one should compare the root loci of

$$1 + \frac{n_0C}{l} \frac{(s+1)}{s(s+0.01)(s+100)} = 0$$

and

$$1 + \frac{n_0C}{l} \frac{(s+1)(s+\lambda)}{s(s+0.01)(s+100)(s+\beta/l)} = 0 \ .$$

In the region of the complex plane $s \ll 100$, s can be ignored with regard to λ and β/l. The root locus is not altered but the gain is reduced. If $n_0C/l \simeq 400$, the transient response for $\beta = 0$ is aperiodic (critical damping, double root near -2) and when $\beta > 0 \{[(\lambda l)/\beta] = 10^{-2}\}$ the dominant roots are $s \simeq -0.025 \pm j\ 0.2$. Therefore, the reactor with delayed neutrons has a weakly damped *oscillatory* response while the reactor without delayed neutrons has a strongly damped aperiodic response.

In conclusion, Professor Tan's analysis precisely shows conditions under which delayed neutrons may make the reactor more stable (or less stable). It does not disprove the statements made previously:

1. A reactor can be unstable, when $\beta > 0$ and asymptotically stable in the small, when $\beta = 0$.
2. In some reactors, the effect of the delayed neutrons is to enhance the oscillatory behavior of the transient response.
3. Consideration of the peaks in the closed-loop transfer function provides information on the transient response (in particular the location of the dominant roots in the complex plane).

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[1]H. B. SMETS, *Nucl. Sci. Eng.*, **25**, 236 (1966).