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Letters to the Editors

Perturbation Theory of Control Rods

Wolfe and Fischer[1] show that the change in ν, the number of neutrons per fission required for criticality, to maintain criticality upon insertion of control rods is given by

$$\frac{\delta\nu}{\nu} = -\frac{\int \phi_2^+ D\nabla_n \phi_2' \, ds}{\int \phi_1^+ \nu \Sigma_f \phi_2' \, dv} \tag{1}$$

within the limitations of two-group diffusion theory. Epithermal fission and epithermal capture of neutrons by the rod are neglected, and the perturbed thermal flux, ϕ_2', is assumed to be zero at the control rod surface; ϕ_1^+ and ϕ_2^+ are the adjoint fluxes in the unperturbed reactor.

The numerator of Eq. (1) represents the net flow of neutron importance into the control-rod surface s. The denominator is the total production of neutron importance.

In two-group diffusion theory, ϕ_2' can be found by solving

$$-\tau\nabla^2\phi_1' + \phi_1' = \frac{\nu\Sigma_f}{\Sigma_{r1}}\phi_2' \tag{2}$$

$$-L^2\nabla^2\phi_2' + \phi_2' = \frac{\Sigma_{sd}}{\Sigma_{a2}}\phi_1' \tag{3}$$

together with the boundary conditions that the thermal flux and the gradient of the fast flux are zero at the control-rod surface. In addition, both ϕ_1' and ϕ_2' should be zero at the reactor-extrapolated outer boundary.

In their first-order theory, Wolfe and Fischer assume the fast flux to be unperturbed on insertion of the control rods. The first-order estimate of the perturbed thermal flux, $\phi_2^{(1)}$, is then found by substituting the unperturbed fast flux, $\phi_1^{(0)}$, in the right-hand side of Eq. (3) and by solving this equation together with the boundary condition of zero thermal flux at the control-rod surface. $\delta\nu^{(1)}$ is then found by making the approximation $\phi_2' = \phi_2^{(1)}$ in Eq. (1).

In a second paper[2], Wolfe and Fischer find the second-order perturbed thermal flux $\phi_2^{(2)}$ by substituting $\phi_2^{(1)}$ in the right-hand side of Eq. (2), solving this equation for $\phi_1^{(1)}$, substituting $\phi_1^{(1)}$ in the right-hand side of Eq. (3), and solving this equation for $\phi_2^{(2)}$; $\delta\nu^{(2)}$ is then found by making the approximation $\phi_2' = \phi_2^{(2)}$ in Eq. (1).

If $C^{(n)}$ is defined by

$$-\int D\nabla_n \cdot \phi_2^{(n)} \, ds = \Sigma_{a2}\, \phi_2^{(0)}(R)\, C^{(n)} , \tag{4}$$

where $\phi_2^{(0)}(R)$ is the average unperturbed thermal flux around the control-rod surface, then the approximation $\phi_2' = \phi_2^{(n)}$ in Eq. (1) gives

$$\frac{\delta\nu^{(n)}}{\nu} = \frac{\phi_2^+(R)\,\Sigma_{a2}\,\phi_2^{(0)}(R)\,C^{(n)}}{\int \phi_1^+ \nu\Sigma_f \phi_2^{(n)} \, dv} , \tag{5}$$

where $\phi_2^+(R)$ is the average unperturbed adjoint thermal flux around the control-rod surface.

Wolfe and Fischer find that $C^{(1)}$ can be identified as the Hurwitz-Roe[3] absorption area C. In addition, they find

$$C^{(2)} = C\,(1-\alpha) , \tag{6}$$

where α is small for $\tau >> L^2$.

Frequently[4] the calculation of the reactivity worth of a regular array of control rods, fully inserted in a finite reactor, can be reduced to the calculation of the worth of a control rod situated at the axis of a circular cylindrical cell, where the dimensions of the cell are such that the flux gradients vanish at the cell outer boundary.

Let us apply the methods of Wolfe and Fischer to determine the worth of a control rod in the cylindrical cell. During iteration n the fast flux $\phi_1^{(n-1)}$ is to be found from $\phi_2^{(n-1)}$ using

$$-\tau\nabla^2\phi_1^{(n-1)} + \phi_1^{(n-1)} = \frac{\nu\Sigma_f}{\Sigma_{r1}}\phi_2^{(n-1)} . \tag{7}$$

Integrating both sides of Eq. (7) over v, the volume between the control-rod surface and the cell outer boundary, and making use of Gauss's theorem, we find

$$\int \phi_1^{(n-1)} \, dv = \frac{\nu\Sigma_f}{\Sigma_{r1}} \int \phi_2^{(n-1)} \, dv , \tag{8}$$

where we have used the boundary conditions of zero fast flux gradient at the control-rod surface and at the cell outer boundary.

The thermal flux $\phi_2^{(n)}$ is now to be found from

$$-L^2\nabla^2\phi_2^{(n)} + \phi_2^{(n)} = \frac{\Sigma_{sd}}{\Sigma_{a2}}\phi_1^{(n-1)} . \tag{9}$$

Integrating both sides of Eq. (9) over v and again using Gauss's theorem

$$-L^2\int \nabla_n \phi_2^{(n)} \, ds + \int \phi_2^{(n)} \, dv = \frac{\Sigma_{sd}}{\Sigma_{a2}} \int \phi_1^{(n-1)} \, dv, \tag{10}$$

where we have used the boundary condition of zero thermal flux gradient at the cell outer boundary.

Now, the unperturbed cell is just critical, so that

$$\frac{\nu\Sigma_f}{\Sigma_r}\frac{\Sigma_{sd}}{\Sigma_{a2}} = 1 . \tag{11}$$

Using Eqs. (4), (8), (10), and (11) we obtain

$$\int \phi_2^{(n)} \, dv = \int \phi_2^{(n-1)} \, dv - c^{(n)}\,\phi_2^{(0)}$$

$$= \phi_2^{(0)} \left[v - \sum_{i=1}^{n} C^{(i)} \right] . \tag{12}$$

Eq. (5) becomes

$$\frac{\delta\nu^{(n)}}{\nu} = \frac{C^{(n)}}{v - \sum_{i=1}^{n} C^{(i)}} . \tag{13}$$

[1] B. WOLFE and D. L. FISCHER, *Nucl. Sci. Eng.* **4**, 785 (1958).
[2] B. WOLFE and D. L. FISCHER, *Nucl. Sci. Eng.* **5**, 5 (1959).
[3] H. HURWITZ and G. ROE, *J. Nucl. Energy* **2**, 85 (1955).
[4] P. GREEBLER, *Nucl. Sci. Eng.* **3**, 445 (1958).

Applying Wolfe and Fischer's Eq. (26), Ref. 1, and Eq. (8), Ref. 2, directly to the case of the rod in the cell, we obtain

$$\frac{\delta\nu^{(n)}}{\nu} = \frac{C^{(n)}}{\nu - C^{(n)}} \;;\; n = 1 \text{ and } 2 \,. \tag{14}$$

Thus Eq. (8), Ref. 2, is algebraically incorrect.

For reactors where $\tau \gg L^2$, the first-order theory, with its assumption of a flat epithermal flux in the cell, can be expected to be quite accurate. Thus for $\tau \gg L^2$, $\delta\nu^{(1)}$ from Eq. (13) (or (14)) is close to the correct result.

Now, for $\tau \gg L^2$, Wolfe and Fischer find that $C^{(2)}$ is closely equal to $C^{(1)}$ so that, whereas from Eq. (14), the process might appear to be converging, it is in fact, from Eq. (13), diverging from the correct result.

The divergence is not due to any inaccuracies in Eq. (13) but to Wolfe and Fischer's approximations made in determining $C^{(n)}$. To find the flux distributions and therefore $C^{(n)}$, Wolfe and Fischer disregard the boundary conditions at the reactor outer boundary or, in this case, at the cell outer boundary, during the iteration process. In solving Eqs. (2) and (3), Wolfe and Fischer retain only the Bessel functions K_n, thus allowing the perturbed flux to approach the unperturbed flux at infinite radius.

If the iteration process were continued, the flux shape would converge to the flux around a control rod situated in an infinite medium. That is, the fluxes would converge to

$$\phi_1' = A\left\{\alpha \ln \frac{r}{B} - \beta \ln \frac{R}{B}\, \frac{K_0(\lambda r)}{K_0(\lambda R)}\right\} \tag{15}$$

$$\phi_2' = A\left\{\ln \frac{r}{B} - \ln \frac{R}{B}\, \frac{K_0(\lambda r)}{K_0(\lambda R)}\right\} \;, \tag{16}$$

where

$$\lambda^2 = \frac{1}{L^2} + \frac{1}{\tau}$$

$$\alpha = \frac{\nu\Sigma_f}{\Sigma_{r1}}$$

$$\beta = \frac{\nu\Sigma_f}{\Sigma_{r1} - D_1\lambda^2}$$

$$B = R \exp\left[\frac{\alpha}{B\lambda R}\, \frac{K_0(\lambda R)}{K_1(\lambda R)}\right] \;.$$

with the constant A normalized so that ϕ_1' and ϕ_2' approach the unperturbed fluxes as r approaches infinity.

For the case of a finite reactor, and especially for the case $\tau \gg L^2$, the converged shape from Eqs. (15) and (16) is nothing like the actual shape of the perturbed flux. Thus for $\tau \gg L^2$, where the fluxes after the first iteration are the correct fluxes, the fluxes after subsequent iterations in converging to the solutions (15) and (16), are in fact diverging from the correct solution.

An attempt to justify Eq. (14) might be made on the grounds that, at least for $\tau \gg L^2$, it gives closely the correct answer for iterations 1 and 2. However, if the iterations are continued past number 2, we see from Eqs. (15) and (16) that finally we obtain flux distributions, which are zero at finite radius but which tend to the unperturbed flux at infinite radius. Thus from Eq. (4), $C^{(\infty)}$ is zero and from Eq. (14), $\delta\nu^{(\infty)}$ is zero.

To conclude, although the first-order perturbation theory by Wolfe and Fischer is valuable for predicting the worth of control rods in reactors where $\tau \gg L^2$, the second-order theory can produce inaccurate results, particularly for reactors where $\tau \gg L^2$. For reactors where τ is not much greater than L^2, the use of the theory to any order does not appear to be justifiable, since, without prior knowledge of the actual form of the perturbed flux, one has no means of determining the iteration number n that will produce $\phi_2^{(n)}$ that is closest to the actual ϕ_2'.

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