Letters to the Editor

On the Eigenvalues of the Stationary and Time-Dependent Boltzmann Equation in Its Integro-Differential and Integral Forms

In a recent Note, Dahl and Sjöstrand[1] performed a numerical analysis of the discrete eigenvalues for the linear stationary monoenergetic Boltzmann equation with anisotropic scattering in multiplying systems with (exact) vacuum boundary conditions. Some aspects of a closely related problem (the determination of the discrete eigenvalues for the time-dependent Boltzmann equation in subcritical systems) were approached by Sjöstrand elsewhere,[2-5] mainly in the nonrigorous frame of the exp(iBr) theory. The author himself noted some contradictions between the various accredited results,[2] and, on the other hand, a series of rigorous aspects of the problem does not seem to be sufficiently known. Therefore, I think it opportune to make some comments related both to the methods applied and to the results obtained.

The discussion essentially involves four eigenvalue equations that, for a slab in the above-mentioned conditions, can be written as

$$(\lambda + 1)\psi(x,\mu) + \mu \frac{\partial \psi}{\partial x}(x,\mu) = \frac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' \quad , \tag{1}$$

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) + \Sigma\psi(x,\mu) = c\Sigma\frac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' \quad , \tag{2}$$

$$\phi(x) = \frac{1}{2}\int_{-a}^{a} E_1[(1+\lambda)|x-x'|]\phi(x')dx' \quad , \tag{3}$$

$$\phi(x) = \frac{c\Sigma}{2}\int_{-a}^{a} E_1(\Sigma|x-x'|)\phi(x')dx' \quad , \tag{4}$$

and that we denote by the abbreviations IDT (integro-differential time-dependent), IDS (integro-differential stationary), IT (integral time-dependent), and IS (integral stationary), respectively. The meaning of the designations is obvious, although in Eqs. (1) and (3), the time dependence is not explicit. Similar equations are written for spherical geometry. The forms of the eigenvalue equations used in Refs. 1 through 4 are somewhat different from ours, e.g., we put $v\Sigma = 1$ and in deriving Eq. (1) we have presumed an exp(λt) [instead of exp($-\lambda v\Sigma t$)] time dependence of the solution.

[1]E. B. DAHL and N. G. SJÖSTRAND, *Nucl. Sci. Eng.*, **69**, 114 (1979).

[2]N. G. SJÖSTRAND, *Nucl. Sci. Eng.*, **63**, 217 (1977).

[3]N. G. SJÖSTRAND, *J. Nucl. Sci. Technol.*, **13**, 81 (1976).

[4]N. G. SJÖSTRAND, *J. Nucl. Sci. Technol.*, **12**, 256 (1975).

[5]N. G. SJÖSTRAND, "Decay Constants for Pulsed Monoenergetic Neutron Systems with Anisotropic Scattering," CTH-RF-28, Chalmers University of Technology (1975).

Now, it is usually claimed[6-9] that the IDT and IDS problems are equivalent for homogeneous systems. Indeed, a *formal* equivalence is easy to evince via the substitutions

$$\Sigma = 1 + \lambda$$

and

$$c = \frac{1}{1+\lambda} \quad .$$

However, the range of the *real* equivalence has not yet been made clear. Some authors[6,7] seem to be aware of this shortcoming, emphasizing the formal aspect of the transformation.

Moreover, another misunderstanding is due to the use, for computational purposes, of the derived corresponding integral forms of the eigenvalue equations instead of the integro-differential forms with which the original problems deal. But, generally, the integral equations are not entirely equivalent with the integro-differential ones, and many extrapolated results are not justified, leading to the abnormalities already noted.[2]

More specifically, we note the following.

1. A rigorous treatment (in a Banach space frame) of the Cauchy initial value problem for the Boltzmann equation in any convex (finite) geometry with vacuum boundary conditions gives a real, algebraically simple eigenvalue for the fundamental mode, not only for monoenergetic scattering (isotropic or not), but also for very general cross sections.[10]

2. For the monoenergetic IDT problem in finite convex (particularly spherical) geometry with vacuum boundary conditions, complex eigenvalues *can* appear for isotropic scattering[11,12] as well as for anisotropic scattering; in any case, their occurrence is not forbidden by any general spectral considerations.

3. For the IDT problem in slab geometry with isotropic scattering and vacuum boundary conditions, all eigenvalues are real, and they are located in the range ($\lambda | -1 < \mathrm{Re}\lambda \leqslant 0$). However, in Ref. 2, Sjöstrand noted that, since the IS equations for spherical and slab geometries are equivalent (up to some convenient and unessential substitutions) and, for the spherical geometry, the eigenvalues can exceed the Corngold

[6]K. M. CASE, F. de HOFFMANN, and G. PLACZEK, "Introduction to the Theory of Neutron Diffusion," Los Alamos Scientific Laboratory (1953).

[7]B. DAVISON, *Neutron Transport Theory*, Oxford University Press, London (1957).

[8]N. G. SJÖSTRAND, *Ark. Fys.*, **15**, 147 (1959).

[9]I. CARLVIK, *Nucl. Sci. Eng.*, **31**, 295 (1968).

[10]N. ANGELESCU and V. PROTOPOPESCU, *Rev. Roum. Phys.*, **22**, 1055 (1977).

[11]K. JÖRGENS, *Comm. Pure Appl. Math.*, **11**, 219 (1958).

[12]J. MARTI, *ZAMP*, **18**, 247 (1967).

limit (in our case, the edge of the continuous spectrum $\mathrm{Re}\lambda = -1$), one can conclude that for the slab, also, the discrete eigenvalues of the IDT problem can exceed this limit. On the other hand, it was stated[13] that these decay constants for the slab cannot exceed the Corngold limit, wherefrom there arises a contradiction.

This contradiction is only virtual, because, *a priori*, the use of IS eigenvalues in computing the IDT ones is not allowed; the spectral analysis of the IDT problem shows that this is not allowed *a posteriori* either. Indeed, for a sphere, it is proved that the integral equation is equivalent with the IDT problem in the whole spectral plane,[11,12] while for the slab, the equivalence is restricted only to the right half-plane, $(\lambda | \mathrm{Re}\lambda > -1)$ (Ref. 14). In the left plane, it is explicitly shown that no eigenvalues (real or complex) can occur and, thus, the IT and, so much the less, the IS equations do not yield eigenvalues with any real significance for the original IDT problem.

4. For anisotropic scattering and slab geometry with vacuum boundary conditions, the question of complex eigenvalues in the IDT problem is not yet solved. This open question must be approached by developing the analysis started by Mika[15] and not by solving some (nonequivalent) IT or IS problems in the left plane $(\lambda | \mathrm{Re}\lambda \leqslant -1)$.

5. As concerns the $\exp(iBr)$ theory, this is formally equivalent with the IDT problem under periodic boundary conditions. Indeed, under periodic boundary conditions,[16] the eigenfunctions have an $\exp(ikx)$ dependence on x; here, k can take a set of discrete values, which lead to different values of λ, according to the eigenvalue equation:

$$1 = \frac{1}{k} \operatorname{arctg} \frac{k}{\lambda + 1} \ ,$$

which is formally identical with the dispersion law,

$$1 = \frac{1}{B} \operatorname{arctg} \frac{B}{\lambda + 1} \ ,$$

obtained from the $\exp(iBr)$ theory. Thus, the role of the buckling parameter, B, which is *fixed* for a given convex body, has no relation with the Fourier parameter k, which, moreover, makes sense only for slab and parallelepipedic geometries. In any case, for both isotropic and anisotropic scattering, the $\exp(iBr)$ theory and the exact solution of the problem under periodic boundary conditions predict a continuous spectrum on the line $(\lambda | \mathrm{Re}\lambda = -1)$, beyond which the dispersion law[2-5] has nothing to do with the IDT problem.

We think that the above remarks clarify some of the aspects existing in the present status of the eigenvalue theory of the Boltzmann equation and can lead to a more adequate approach to the complex eigenvalues problem for anisotropic scattering in slab geometry with vacuum boundary conditions, the only one toward which, in the author's opinion, some effort might be worthy.

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[13] M. M. R. WILLIAMS, *The Slowing Down and Thermalization of Neutrons*, North Holland Publishing Company, Amsterdam (1966).

[14] J. LEHNER and G. M. WING, *Comm. Pure Appl. Math.*, **8**, 217 (1955).

[15] J. MIKA, *J. Math. Phys.*, **7**, 833 (1966).

[16] N. ANGELESCU, N. MARINESCU, and V. PROPTOPOPESCU, *Transp. Theory Stat. Phys.*, **5**, 115 (1976).

Comments on "On the Eigenvalues of the Stationary and Time-Dependent Boltzmann Equation in Its Integro-Differential and Integral Forms"

The Letter by Protopopescu[1] clarifies many obscure points regarding the existence of eigenvalues to the transport equation. It contains the information I hoped to provoke through an earlier Note.[2]

An important fact in this connection is, of course, that a solution to the integro-differential Boltzmann equation gives the angular-dependent neutron flux density,

$$\psi(x,\mu) = \phi_0(x) + 3P_1(\mu)\phi_1(x) + \dots \ , \qquad (1)$$

whereas the integral equation in its usual form only contains the total neutron flux density $\phi_0(x)$. One of Protopopescu's statements (under his point 3) can therefore be expressed as follows: Even though the integral equation for $\phi_0(x)$ in an infinite slab has eigenvalues beyond the Corngold limit in the case of isotropic scattering, these eigenvalues have no relevance for the complete solution of Eq. (1), which cannot exist in this region.

Some observations in our numerical work[3,4] corroborate the stringent results presented by Protopopescu and shed some further light on the problem. The equivalence between the time-dependent and stationary cases implies that a decay constant beyond the Corngold limit corresponds to a negative total cross section in the stationary case. Since the total cross section enters as a factor into the arguments of the exponential integrals in the matrix elements,[3,5] a negative total cross section means that the exponential integrals diverge. In the case of an infinite slab with vacuum boundary conditions, we have observed the following:

1. Integral equation for ϕ_0, isotropic scattering. The expoential integrals cancel in the matrix elements for the odd harmonic modes. As stated in the earlier Note,[2] this means that eigenvalues can be obtained numerically for ϕ_0 above the Corngold limit.

2. Integral equation for ϕ_0, anisotropic scattering. In this case, it has not been possible to establish an integral equation for the odd harmonic modes of ϕ_0, only an equation coupled to the neutron current density ϕ_1.

3. Integral equation for ϕ_1, isotropic and anisotropic scattering. The exponential integrals seem to persist in the matrix elements. For isotropic scattering and odd harmonic modes, however, the structure of the matrix is such that the exponential integrals do not enter into the calculation of the eigenvalues, only of the eigenvectors. This means that no complete solution of the type of Eq. (1) can be found beyond the Corngold limit. For anisotropic scattering or even harmonic modes, the eigenvalues depend on the exponential integrals.

Regarding the $\exp(iBr)$ theory discussed by Protopopescu, it might be of interest to recall its use in the interpretation of time-dependent experiments with the pulsed neutron source

[1] V. PROTOPOPESCU, *Nucl. Sci. Eng.*, **71**, 228 (1979).

[2] N. G. SJÖSTRAND, *Nucl. Sci. Eng.*, **63**, 217 (1977).

[3] E. B. DAHL and N. G. SJÖSTRAND, *Nucl. Sci. Eng.*, **69**, 114 (1979).

[4] E. B. DAHL, Private Communication (1979).

[5] I. CARLVIK, *Nucl. Sci. Eng.*, **31**, 295 (1968).

[6] K. H. BECKURTS and K. WIRTZ, *Neutron Physics*, Springer-Verlag, Berlin (1964).