Letters to the Editor

Comment on the Nonhyperbolicity of Two-Phase Flow Equations

In a recent Note, Lyczkowski[1] considered why the two-phase flow equations having equal phase pressure are not hyperbolic. Lyczkowski[1] extended the analysis of a single fluid in an elastic pipe to the set of two-phase flow equations. He showed that if the phase flow area for each phase is taken to be a function of a single pressure, complex characteristics can occur. Since his characteristic analysis has not taken into account the equal pressure constraint, his explanation needs some modifications. In this Letter, we want to show that if the phase flow area is a function of a single pressure, the two-phase flow equations are physically hyperbolic. However, if the phase flow area is not explicitly constrained by pressure or other variables, the two-phase flow equations, which contain no other static and dynamic forces, are not totally hyperbolic.

First, we review the characteristic analysis of the two-phase equations considered by Lyczkowski.[1] Without loss of generality, we remove the need to consider the energy equations by assuming a constant enthalpy flow such that $\rho_k = \rho_k(p_k)$, where the subscript k denotes phase k, ρ and p are the cross-sectional-averaged density and pressure. The continuity and momentum equations for phase k are

$$\frac{\partial}{\partial t}(A\alpha_k\rho_k) + \frac{\partial}{\partial z}(A\alpha_k\rho_k u_k) = 0 \tag{1}$$

and

$$\frac{\partial}{\partial t}(A\alpha_k\rho_k u_k) + \frac{\partial}{\partial z}(A\alpha_k\rho_k u_k^2) + A\alpha_k\frac{\partial p_k}{\partial z} = 0 \ , \tag{2}$$

where A is the pipe cross-section area, α_k is the phase volume fraction, and u_k is the phase velocity. In Lyczkowski's work, he set

$$A_k = A\alpha_k \ .$$

The coupling between two phases occurs through the phase pressure p_k and phase volume fraction α_k. When both phase pressures are equal ($p_k = p$), Eq. (2) corresponds to Eq. (19) of Lyczkowski.[1]

By assuming that α_k is a function of the phase pressure p_k and the total flow area A is constant, the characteristics of Eqs. (1) and (2) without the equal pressure constraint are

$$\lambda_{1,2} = u_g \pm \frac{a_g}{\left(1 + \dfrac{\rho_g a_g^2}{\alpha_g}\dfrac{d\alpha_g}{dp_g}\right)^{1/2}} \tag{3}$$

and

$$\lambda_{3,4} = u_f \pm \frac{a_f}{\left(1 + \dfrac{\rho_f a_f^2}{\alpha_f}\dfrac{d\alpha_f}{dp_f}\right)^{1/2}} \ , \tag{4}$$

where the subscripts g and f denote gas (or vapor) and liquid phases, respectively, and the sound speed of phase k is defined by

$$a_k^2 = \left(\frac{\partial\rho_k}{\partial p_k}\right)^{-1} \ ; \quad k = g,f \ . \tag{5}$$

Equations (3) and (4), which correspond to Eqs. (25a) and (25b) of Lyczkowski,[1] are obtained without restricting the two-phase pressure relationships. If the phase volume fraction is a function of the phase pressure, the constraint on $\alpha_k(\alpha_g + \alpha_f = 1)$ requires that $p_g = p_f$. Therefore, Eqs. (3) and (4) are not the characteristics of Eqs. (1) and (2) when two-phase pressures are equal.

When two-phase pressures are equal and the phase flow area or volume fraction is a function of pressure, the governing equations become

$$\frac{\partial}{\partial t}(A\alpha_g\rho_g + A\alpha_f\rho_f) + \frac{\partial}{\partial z}(A\alpha_g\rho_g u_g + A\alpha_f\rho_f u_f) = 0 \tag{6}$$

and

$$\frac{\partial}{\partial t}(A\alpha_k\rho_k u_k) + \frac{\partial}{\partial z}(A\alpha_k\rho_k u_k^2) + A\alpha_k\frac{\partial p}{\partial z} = 0 \ . \tag{7}$$

The characteristic velocities are given approximately by

$$\lambda_{1,2} = u \pm a \tag{8}$$

and

$$\lambda_3 = v \ , \tag{9}$$

where

$$u = \frac{\alpha_g u_g + \alpha_f u_f}{z} + \frac{1}{2}\,\frac{u_g C_g + u_f C_f}{C_g + C_f} \ ,$$

$$a^2 = \left[\frac{\alpha_g}{a_g^2} + \frac{\alpha_f}{a_f^2} - (\rho_f - \rho_g)\frac{d\alpha}{dp}\right]^{-1} \ ; \quad \alpha = \alpha_g \ ,$$

$$v = \alpha_g u_f + \alpha_f u_g \ ,$$

and

$$C_k = \frac{\alpha_k}{a_k^2} + \rho_k\frac{d\alpha_k}{dp} \ .$$

The hyperbolic condition is given by

$$\frac{d\alpha}{dp} < \frac{1}{\rho_f - \rho_g}\left(\frac{\alpha_g}{a_g^2} + \frac{\alpha_f}{a_f^2}\right) \ . \tag{10}$$

[1] R. W. LYCZKOWSKI, *Nucl. Sci. Eng.*, **76**, 246 (1980).

The characteristics λ_1 and λ_2 represent the sound-wave characteristics along which small-amplitude pressure disturbances propagate. The third characteristic, λ_3, represents a slow-wave characteristic along which the velocity-difference or phase-slip disturbance propagates.

If $d\alpha/dp < 0$, $\lambda_{1,2}$ will be real. Consider the case of thermal equilibrium, then the vapor void fraction α is a function of the pressure and mixture specific enthalpy h. For a flow with constant enthalpy ($dh = 0$), α is a function of pressure only. In this case, $d\alpha/dp$ is given by

$$\frac{d\alpha}{dp} = -\frac{\frac{\rho}{\rho_g} F + \frac{\chi\rho}{\rho_g^2 a_g^2} - \frac{x}{\rho_g}\left(\frac{\alpha_g}{a_g^2} + \frac{\alpha_f}{a_f^2}\right)}{1 + \frac{x}{\rho_g}(\rho_f - \rho_g)} , \tag{11}$$

where

$$F = \frac{1}{h_{gs} - h_{fs}} \frac{dh_{fs}}{dp} + \frac{h - h_{fs}}{(h_{gs} - h_{fs})^2}\left(\frac{dh_{gs}}{dp} - \frac{dh_{fs}}{dp}\right),$$

$$h = xh_g + (1 - x)h_f ,$$

$$\rho = \alpha_g\rho_g + \alpha_f\rho_f ,$$

$$x = \frac{h - h_{fs}}{h_{gs} - h_{fs}} ,$$

and

$$\alpha = \frac{\chi\rho}{\rho_g} , \tag{12}$$

and the subscript s denotes saturation line. Here, h is the specific enthalpy of phase k and x is the static quality.

Using the fact that $a_g < a_f$, $\rho \geqslant \rho_g$, and $F > 0$, it is clear that $d\alpha/dp < 0$. Therefore, the two-phase flow equations are always hyperbolic for the case of thermal equilibrium. Since the equation set for the homogeneous nonequilibrium model is always hyperbolic, we would expect that $d\alpha/dp < 0$ is true also for the thermal nonequilibrium case. Hence, if the phase flow area is taken to be a function of phase pressure, the two-phase flow equation set is always hyperbolic for the case of equal phase pressures.

However, if the phase flow area is not explicitly dependent on pressure and/or other variables (i.e., there is no constraint on α), the characteristics for Eqs. (1) and (7) are

$$\lambda_{1,2} = u \pm a \tag{13}$$

and

$$\lambda_{3,4} = v \pm w , \tag{14}$$

where

$$u = \frac{\alpha_g\rho_f u_g + \alpha_f\rho_g u_f}{\alpha_f\rho_g + \alpha_g\rho_f} ,$$

$$a^2 = \frac{\alpha_f\rho_g + \alpha_g\rho_f}{\frac{\alpha_f\rho_g}{a_f^2} + \frac{\alpha_g\rho_f}{a_g^2}} ,$$

$$v = \frac{\alpha_f\rho_g u_g + \alpha_g\rho_f u_f}{\alpha_g\rho_f + \alpha_f\rho_g} ,$$

and

$$w = i\,\frac{(u_g - u_f)(\alpha_g\alpha_f\rho_g\rho_f)^{1/2}}{\alpha_g\rho_f + \alpha_f\rho_g} .$$

It is clear that the equation set is not hyperbolic unless $u_g = u_f$. The equation set can be hyperbolic only when other static and/or dynamic forces are included in the momentum equations. For example, the gravity force, interfacial pressure term, surface tension, viscous term, and the virtual mass force, which constrain the phase pressure, void fraction, or phase velocities, will render the equation set hyperbolic. As a result of the constraints on phase pressure, void fraction, and phase velocities, which occur physically, additional couplings between two phases occur.

It can be concluded that the constraint on the phase pressure results in a change in the sound-wave characteristics (i.e., two pairs of sound waves become a pair of sound wave) and constraints on volume fraction and phase velocities modify the slow-wave characteristics of the two-fluid model. Complex characteristics of the equation set result when the physical constraints on the volume fraction and phase velocities are ignored.

W. S. Liu

Ontario Hydro
Nuclear Studies and Safety Department
Toronto, Ontario, Canada

February 4, 1981

Reply to "Comment on the Nonhyperbolicity of Two-Phase Flow Equations"

The analysis[1] may contain elements of a contradictory nature for the following reasons.

1. Liu used a different equation-of-state, $\rho_k = \rho_k(p)$, to arrive at his Eqs. (6) and (17). This reduced the order of the system from four to three and he obtained different characteristics.

2. The quantity $d\alpha/dP$ is not always negative. The most obvious situation is the decompression of an initially saturated vapor that causes condensation to occur (Wilson cloud chamber effect). The value $a_f^{-2} = d\rho_f/dP$ is negative along the saturation line.

3. Lyczkowski et al.[2] found that Eqs. (1) and (7) produce a characteristic polynomial which does not factor. If $a_f^{-2} = a_g^{-2} = 0$, then Eq. (14) results, but λ_{12} are infinite.

Robert W. Lyczkowski

Illinois Institute of Technology
Chicago, Illinois 60616

February 20, 1981

[1] W. S. LIU, *Nucl. Sci. Eng.*, **78**, 305 (1981).

[2] ROBERT W. LYCZKOWSKI, DIMITRI GIDASPOW, CHARLES W. SOLBRIG, and E. D. HUGHES, *Nucl. Sci. Eng.*, **66**, 378 (1978).