$$\omega\tau^* \ell n\, C = -\omega t + \frac{2}{\sqrt{(1-\epsilon_1)^2-\epsilon_2^2}}\left[\tan^{-1}\left\{\frac{(1-\epsilon_1)\tan\frac{\omega t}{2}-\epsilon_2}{\sqrt{(1-\epsilon_1)^2-\epsilon_2^2}}\right\}+\right.$$

$$\left.-\tan^{-1}\left\{\frac{-\epsilon_2}{\sqrt{(1-\epsilon_1)^2-\epsilon_2^2}}\right\}\right] \qquad (10)$$

The value of $C(t)$ can be written more simply when $\omega t = (2n+1)\pi$. If we restrict consideration of the solution to only those values of t when the above equation is satisfied,

$$\omega\tau^* \ell n\, C = -\omega t + \frac{\omega t}{\sqrt{(1-\epsilon_1)^2-\epsilon_2^2}} \qquad (11)$$

Substitution of Eq. (11) into the equation for neutron density, Eq. (7), then yields

$$N(t) = \frac{1}{1-\epsilon_1-\epsilon_2 \sin\omega t}\exp\left\{\frac{t}{\tau^*}\left[\frac{1}{\sqrt{(1-\epsilon_1)^2-\epsilon_2^2}}-1\right]\right\} \qquad (12)$$

If ϵ_1 is chosen equal to zero (i.e., for reactivity oscillation about a zero mean value), the neutron flux can be approximated by

$$N(t) = [1+\epsilon_2 \sin\omega t]\exp\left\{\frac{\epsilon_2^2}{2}\frac{t}{\tau^*}\right\}. \qquad (13)$$

Hence, the mean power drift is exponential with an e- folding time of $2\tau^*/\epsilon_2^2$ and has a second-order dependence on the reactivity perturbation. It should be noted that the result is independent of the oscillation frequency, ω.

Evidently, the drift can be reduced to zero by setting

$$(1-\epsilon_1)^2-\epsilon_2^2 = 1$$

or

$$\epsilon_1 = 1-\sqrt{1+\epsilon_2^2} \approx -\frac{\epsilon_2^2}{2}$$

Although the simplicity of the model precludes rigorous application of the solution, the results are in agreement with physical intuition and show the important role of the delayed neutrons in causing the mean power drift during an oscillation-experiment.

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A More Straightforward Use of Variational Principles with Boundary Conditions

Pomraning and Clark[1] recently added boundary conditions to a variational principle for solutions to the transport equation[2] (the additional terms, although described as having to be guessed, were derived from a general procedure independently by Selengut[3]). Their treatment of one-group diffusion theory for a homogeneous slab will now be reviewed for the special situation in which scattering is isotropic and no neutrons enter the medium from outside its surfaces.

For a slab extending from $z=a$ to b, a functional is defined

$$F[\phi(z,\mu)] \equiv \int_a^b dz \int_{-1}^{1} d\mu\, [\phi^* H\phi - S\phi^* - T\phi] +$$

$$+\int_0^1 d\mu\, \mu\, \phi^*(a,\mu)\, \phi(a,\mu) - \int_{-1}^0 d\mu\, \mu\, \phi^*(b,\mu)\, \phi(b,\mu).$$

The operator H is given by

$$H\phi(z,\mu) \equiv \mu \frac{\partial\phi}{\partial z} + \phi - \frac{c}{2}\int_{-1}^{1} d\mu'\, \phi(z,\mu'),$$

where z is taken in units of total mean free paths. The symbol c is the number of secondaries per collision, S is the (given) source, and

$$\phi^*(z,\mu) \equiv \phi(z,-\mu),$$
$$T(z,\mu) \equiv S(z,-\mu).$$

Imposing the condition that F be stationary with respect to arbitrary variations in ϕ is equivalent to making ϕ equal the solution to the transport equation:

$$H\phi(z,\mu) = S(z,\mu) \text{ and}$$
$$\phi(a,\mu) = 0 \text{ for } \mu > 0.$$
$$\phi(b,\mu) = 0 \text{ for } \mu < 0.$$

We now further specialize to the situation in which S is isotropic:

$$S(z,\mu) = T(z,\mu) = \frac{1}{2} S_0(z).$$

The authors make a P-1 approximation,

$$\phi(z,\mu) = \frac{1}{2}\phi(z) + \frac{3}{2}\mu J(z),$$

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[1] G. C. POMRANING and M. CLARK, Jr., *Nucl. Sci. Eng.* **16**, 147-154 (1963).

[2] D. S. SELENGUT, *Hanford Quarterly Report*, HW-59126, 89-124, Richland, Wash., (1959).

[3] D. S. SELENGUT, *Trans. Am. Nucl. Soc.*, **5**, 1 (1962). p. 40.

and form the 'reduced Lagrangian' (since it will be impossible to make this quantity stationary for all arbitrary variations in ϕ and J, the term 'reduced Lagrangian' is really a misnomer) as a functional of the scalar flux $\phi(z)$ and current $J(z)$,

$$F[\phi,J] = \int_a^b dz \left[\frac{1}{2}\phi\frac{dJ}{dz} - \frac{1}{2}J\frac{d\phi}{dz} + \frac{1-c}{2}\phi^2 - \frac{3}{2}J^2 - S_0\phi\right] + \left[\frac{\phi^2(a)}{8} - \frac{9}{16}J^2(a)\right] + \left[\frac{\phi^2(b)}{8} - \frac{9}{16}J^2(b)\right].$$

Requiring the first variations of this functional to vanish produces the familiar differential equations of 'diffusion theory',

$$\frac{d\,J(z)}{dz} + (1-c)\,\phi(z) = S_0(z),$$

$$J(z) = -\frac{1}{3}\frac{d\,\phi(z)}{dz},$$

along with a condition to complete the specification of ϕ and J,

$$\left[\frac{1}{4}\phi(a) + \frac{1}{2}J(a)\right]\delta\phi(a) - \left[\frac{1}{2}\phi(a) + \frac{9}{8}J(a)\right]\delta J(a) + \left[\frac{1}{4}\phi(b) - \frac{1}{2}J(b)\right]\delta\phi(b) + \left[\frac{1}{2}\phi(b) - \frac{9}{8}J(b)\right]\delta J(b) = 0.$$

This letter is primarily concerned with the remaining steps taken by the authors. They note that making the coefficient of $\delta\phi(a)$ vanish implies the boundary condition that

$$\Gamma \equiv -\frac{J(a)}{\phi(a)} = \frac{1}{2},$$

while the vanishing of the coefficient of $\delta J(a)$ implies the contradictory condition that Γ equals $\frac{4}{9}$. At the other boundary, the same contradictory values appear for $J(b)/\phi(b)$, which is therefore also called Γ. Obviously, some restrictions must be placed on the variations allowed! The authors' reaction was to eliminate J at each boundary in favor of Γ, which was then held stationary as ϕ was varied. The resulting coefficient of $\delta\phi$ at each boundary vanished if

$$\Gamma = \pm\sqrt{\frac{2}{9}}.$$

For physical reasons, only the positive root was accepted.

This value gives an extrapolation length

$$d \equiv 1/(3\Gamma) = 0.7071,$$

which is very close to 0.7104, the value for the exact Milne problem for a non-absorbing medium. In contrast, values of Γ equal to $\frac{1}{2}$ from Marshak boundary conditions and $1/\sqrt{3}$ from Mark boundary conditions, yield values of d equal to 0.6667 and 0.5773 respectively. The conclusion was that a method had been found for producing considerable improvements over the results of existing procedures.

However, certain features of this analysis and the later extensions[4-6] raise some questions:

1) *Why were the restrictions on the variations of the flux components put in the form of linear couplings at each boundary?* The only reason given for holding Γ constant was that it would remove the dilemma of having contradictory results, but so would an infinitude of other restrictions. The step certainly does not follow from the linearity of H. Arbitrarily holding Γ constant would seem merely to lead to a fortuitous compromise between the contradictory values implied by independent variations (neither of which is very bad). If simplicity was desired, holding $J(a)$ and $J(b)$ stationary would have been simpler; as will be seen, even that step often turns out to be more accurate as well.

2) *Why did the variational principle fail to provide a unique result?* Physical reasoning was required to eliminate the negative square root of Γ, and the occurrence of extraneous roots was even more pronounced in later work[4,6]. Yet, since the variational principle is equivalent to a specification of all the conditions that the solution must fulfill, it should certainly lack no information needed to complete the determination of any reasonable approximate function. Note that this matter is quite apart from relying on intuition to choose the approximate form of the function before it goes into the variational principle.

3) *Was it reasonable for the variational principle to imply that the same rigid boundary condition should be used for such an extensive vdriety of situations?* According to the procedure reviewed here, the variational principle furnishes a single fixed value of the extrapolation length, which is to be used in diffusion theory for every bare slab, regardless of its thickness, source distribution, and the number of secondaries per collision. An ability to adjust for the special properties of individual situations would be a major advantage over rigid boundary conditions like Mark's or Marshak's. A variational principle would seem ideal for producing this feature. More accurate calculations do[3,6].

4) *Did comparisons with results from the non-absorbing Milne problem provide a conclusive test?* Evidently, the linear boundary coefficients

[4] G. C. POMRANING and M. CLARK, Jr., *Nucl. Sci. Eng.* **17**, 227-233 (1963).
[5] G. C. POMRANING, *Nucl. Sci. Eng.* **18**, 528-530 (1964).
[6] G. C. POMRANING *Ann. Phys.* **27**, 193 (1964).

obtained from the variational principle are to be used for all bare slabs, and in the P-3 case[5,6] they were in fact specifically recommended for many types. These boundary conditions were declared 'improved' (over Marshak's e.g.) on the basis of how closely they reproduced the extrapolation length of the non-absorbing Milne problem. If such a criterion could actually decide universally best boundary conditions, one could simply adopt those given by the exact solution to the non-absorbing Milne problem! However, a forthcoming example supports the opposite conjecture: rigid boundary conditions specialized to one situation (even fortuitously) are likely to be bad for others. For situations other than source-free infinite media, the Pomraning-Clark rigid conditions were not shown to be better than those of Mark or Marshak. For one example, they will be shown to be worse.

5) *Apart from the question as to which physical situations should have been considered, did the quantities examined for error provide a conclusive test?* In some situations[1,4-6], the accuracy of the extrapolation length at a free boundary was tested; in others[6], it was the capture fraction of neutrons incident from the outside. There is no reason to expect that the accuracy of any single arbitrary quantity like these can generally be used to determine the accuracy of the flux at all positions. Indeed, since an approximation of given order has limited accuracy, a change from one reasonably good boundary condition to another would improve some properties of the flux necessarily at the expense of others; the accuracy of the functional itself is all that the variational principle is known to emphasize.

6) *Since this procedure yields results different from those of the 'direct method' when identical quantities are to be determined, has an improvement over this standard old technique been established?* In the 'direct method', the parameters of an explicit trial function are determined by requiring the first partial derivatives of the resulting functional with respect to each to vanish. The treatment of boundary conditions is straightforward. Now if a particular trial function happens to be the most general solution to the Euler-Lagrange differential (e.g., diffusion) equations, the direct method yields parameters differing from those of Pomraning and Clark. If their procedure is actually better, an attempt should be made to extend it to the general direct method.

Most of these questionable points seem to have resulted from a belief that the selection of a particular solution to the differential equations should be based on linear relations between the flux components at each boundary. However, a solution could be selected by many other means. For an alternate condition, it is now suggested that the functional of the most general solution to the differential equations be made stationary with respect to its arbitrary constants. The difficulties mentioned will then be eliminated.

For example, the general solution of the diffusion equation is

$$\phi(z) = \phi_P(z) + A \cosh Kz + B \sinh Kz,$$

$$-3J(z) = \phi_P'(z) + AK \sinh Kz + BK \cosh Kz,$$

where $K \equiv \sqrt{3(1-c)}$, $\phi_p(z)$ is a (known) particular solution to the inhomogeneous diffusion equation, and A and B are the desired coefficients of the homogeneous solutions. Since the integral term of the variations of the functional automatically vanishes, merely insert the above expressions and their variations,

$$\delta\phi(z) = \delta A(\cosh Kz) + \delta B(\sinh Kz),$$

$$-3\,\delta J(z) = \delta A(K \sinh Kz) + \delta B\,(K \cosh Kz),$$

evaluated at $z = a$ and b, into the expression for the vanishing of boundary condition variations. Then evaluate A and B from the condition that the coefficients of δA and δB vanish.

For many important situations, $a = -b$, $S_0(z)$ is symmetric about $z = 0$, and as a result, $B = 0$. For example, for each source term of the form,

$$S_0(z) = \cos \beta z \text{ for } -b \le z \le b,$$

the coefficient A has a corresponding term,

$$\frac{\left(\frac{2}{3}\cos\beta b - \frac{1}{2}\beta\sin\beta b\right) K\sinh Kb + \left(\frac{2}{3}\beta\sin\beta b - \cos\beta b\right)\cosh Kb}{\left(1-c+\frac{1}{3}\beta^2\right)\left(\cosh^2 Kb - \frac{1}{2}K^2\sinh^2 Kb\right)}$$

In order to test the accuracy of this method, we now consider a situation with the interesting property of being analytically solvable both by transport theory and diffusion theory. Let $a = 0$ for a non-absorbing ($c = 1$) slab containing a symmetrically distributed source decreasing monotonically from the surfaces to the center,

$$S_0(z) = \frac{1}{2}\left[E_2(z) + E_2(b-z)\right],$$

where

$$E_n(x) \equiv \int_0^1 \mu^{n-2} e^{-\frac{x}{\mu}}\, d\mu$$

is a tabulated function[7]. The exact vector flux from transport theory is

[7] K. M. CASE, F. de HOFFMAN and G. PLACZEK, *Introduction to the Theory of Neutron Diffusion*, p. 153. Los Alamos Scientific Laboratory, Los Alamos, New Mexico, (1953).

$$\phi_T(z,\mu) = \frac{1}{2}\left(1 - \exp - \frac{z}{\mu} + \frac{b}{\mu}\right) \text{ for } \mu < 0$$

$$= \frac{1}{2}\left(1 - \exp - \frac{z}{\mu}\right) \text{ for } \mu > 0,$$

and the resulting scalar flux is

$$\phi_T(z) \equiv \int_{-1}^{1} \phi_T(z,\mu)\, d\mu = 1 - \frac{1}{2}\left[E_2(z) + E_2(b-z)\right] .$$

The diffusion-theory solution with the same source is

$$\phi_D(z) = A - \frac{3}{2}\left[E_4(z) + E_4(b-z)\right],$$

$$J_D(z) = \frac{1}{2}\left[E_3(b-z) - E_3(z)\right],$$

where a term Bz was dropped from ϕ_D by symmetry. Since A does not appear in $J_D(z)$,

$$\delta J_D = 0,$$

and $\Gamma = \frac{1}{2}$, the Marshak value.

Note that some properties here are a special consequence of source symmetry and the absence of absorption. As illustrated in the previous example, the results for other circumstances do not agree with Marshak's, and there is a greater dependence on the slab and source properties. Here the diffusion theory expression for the current is exact, and the variational principle duplicates the Marshak condition in stressing the importance of this quantity.

Now the results will be tested numerically for the situation in which $b \to \infty$. For different types of boundary conditions,

A = 0.9330 by Mark

= 1.0000 by Marshak

= 1.0000 by present work

= 1.0303 by Pomraning and Clark

= 1.0328 by Milne

Figure 1 shows the exact scalar flux along with those calculated by diffusion theory with these coefficients. While the Mark conditions are clearly the worst, the example shows that those of Pomraning or Milne (which are nearly indistinguishable) are not always 'better' than those of Marshak. The curves also demonstrate the dangers of judging a boundary condition by the accuracy of an arbitrary property, since no boundary conditions produce the best flux at all positions: the flux of Milne is the most accurate at positions that are of the order of a mean free path from the surface, but only Marshak's and our suggested boundary conditions produce exact values at the surface and throughout an infinite interior length.

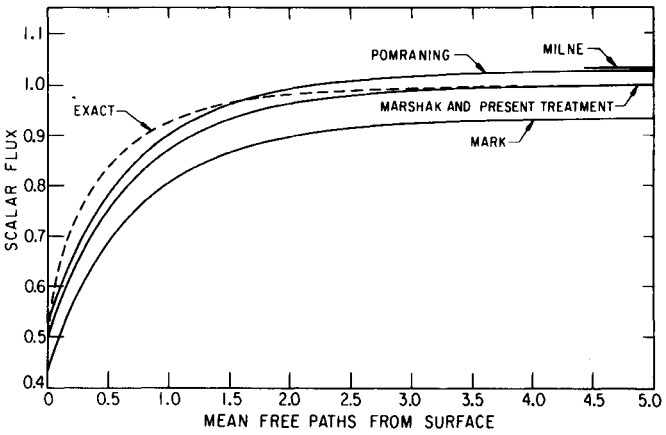

Fig. 1. Scalar flux from transport theory and diffusion theory with various boundary conditions. The medium is a very thick nonabsorbing slab with a source distribution gradually decreasing from the surfaces to the center.

The variational principle yields an entire flux distribution, which obviously can be used directly for many purposes. Its only exceptional feature, however, is the resulting accuracy imparted to the functional itself, which can be made the flux weighted average of any quantity desired[2]. Examining this quantity, therefore, would seem to be the most reasonable way to test the power of the variational principle. Moreover, since the sole mechanism of this method is to make an approximate functional resemble the exact one, its accuracy would also provide some indication of the soundness of conflicting mathematical procedures.

The general expressions show that for the adjoint source in this letter, the exact

$$F[\phi_T(z,\mu)] = -\frac{1}{2}\int_a^b \phi_T(z)\, S_0(z)\, dz,$$

and the approximate

$$F[\phi_D, J_D] = -\frac{1}{2}\int_a^b \phi_D(z)\, S_0(z)\, dz + \frac{\phi_D^2(a)}{8} - \frac{9}{16}J_D^2(a) + \frac{\phi_D^2(b)}{8} - \frac{9}{16}J_D^2(b) .$$

For the numerical case considered here,

$$F[\phi_T(z,\mu)] = \frac{1}{4}\int_0^\infty E_2^2(z)dz - \frac{1}{4} = -\frac{1}{12}(1 + 2 \ln 2),$$

and

$$F[\phi_D, J_D] = \frac{1}{4}I + \frac{1}{4}A^2 - \frac{1}{2}A - \frac{1}{128},$$

where

$$I = 3\int_0^\infty E_2\, E_4\, dz = \frac{11 - 12 \ln 2}{10}.$$

The final result is that the error in the reduced functional is

$$F[\phi_D, J_D] - F[\phi_T] = 0.0081 + \frac{1}{4}(A-1)^2 .$$

Thus, the value $A = 1$, given by the treatment suggested here, yields the best possible functional that can be obtained by diffusion theory! For close values, the boundary terms make the functional much more accurate than the expression for $F[\phi_T(z,\mu)]$ with $\phi_T(z)$ merely replaced by $\phi_D(z)$.

It is now incumbent on this letter to justify a more general basis for the suggested procedure and to define the conditions for its validity. It would also be desirable to alleviate some former anxiety that "the variational method has given us too much information"[1] and "the reason that this trouble arises is apparently a peculiarity in the application of the variational method to the transport equation"[6]. Rather, it is satisfying to find that a straightforward use of the variational principle always produces exactly enough information to determine all the unspecified properties of the trial function.

The functional is stationary for all variations in its argument only when they are about the exact solution. Therefore, if a trial function incorporates any approximation, there is always a possibility of imposing too much stationariness about it, and results from such action must be contradictory. A straightforward procedure, however, would be to require stationarity at each step only for variations that preserve all previously established properties. When a quantity is unknown, the arbitrary variations made on it will then be completely uncoupled with others, and its determination will be equivalent to utilizing a (reduced) variational principle. The uniqueness and reliability generally available from variational principles will therefore be established automatically. On the other hand, one could attempt to make the original functional stationary for variations of a quantity about some value already fixed in the trial function. Then a danger is that such stationarity may not be very dependent on the quantities that were meant to be determined.

For example, say a P-1 approximation is adopted. The variational principle without boundary conditions then yields the usual P-0 and P-1 equations when stationarity is imposed for variations of the two retained Legendre components. These variations are within the constraints of the P-1 approximation; however, a possible alternate way of imposing stationarity about a P-1 approximation would be to do so for variations of any higher Lengendre component from its zero value. For any single component, the usual P-equation associated with it results, but with only the P-0 and P-1 terms not zero. Thus, variations of the P-2 component imply results that are incompatible with the P-0 and P-1 equations; experience is that the latter are more reliable. Variations of still higher components yield no information at all.

In the straightforward procedure suggested, it does not matter whether an approximate condition adopted in the trial function came directly from outside the variational principle or from a partial application of it. A reduced variational principle can be formed in either case. When boundary conditions are included in the functional and a P-1 approximation is made, stationarity is possible under enough variations to imply the differential equations of diffusion theory. The method suggested immediately adopts the differential equations as a constraint on all further variations, and the last of the successively reduced variational principles is only for the arbitrary constants in the general solution.

Since an argument function obtained through the suggested method does not render the functional stationary with respect to variations in the approximate conditions, whatever was the source of these conditions must also be the provider of their justification. If all the conditions are quite accurate, the consequences of imposing stationarity will likewise be accurate, but the results are typically so sensitive to the validity of these constraints, that poor assumptions can rapidly lead to disaster.

For example, if a non-absorbing slab contains an asymmetric source and is $1/\sqrt{2}$ mean free paths thick, the antisymemtric solution to the homogeneous diffusion equation will be assigned an infinite coefficient[8]. This combination of source asymmetry and slab thinness thoroughly destroy the requisite validity of diffusion theory. It is certainly plausible that an approximation specialized to one type of situation would be exceptionally poor for the extreme opposite.

Conversely, if rigid boundary conditions yield an accurate result from a faulty use of diffusion theory, they are likely to be hazardous for proper uses. For isotropic and linearly anisotropic beams directed into a semi-infinite medium[6], capture fractions from Pomraning's boundary conditions were very accurate but high in both cases. Results from Marshak boundary conditions were rather inaccurate, but since they were high in the first situation and rather low in the second, they must be better than Pomraning's for beams with some intermediate angular distributions. S-8 calculations[9] show that these intermediate cases are

[8] G. C. POMRANING (private communication).

[9] Furnished privately by T. PERKINS, Aerojet General Nucleonics Corp.

just the ones for which diffusion theory is best, for it describes the flux near the surface quite poorly for the two sources considered, but with an (area normalized) angular and spatial distribution between those of the two actual fluxes (all observations are consistent with diffusion theory being best for a source closer to the isotropic of the two). Thus, the very data used by Pomraning to demonstrate that his method is "superior" indicate that it compares worst when the variational principle should be at its best.

When any calculational method applied to a restricted set of situations makes some arbitrary gross quantity remarkably more accurate than the detailed distributions or any theoretical reason would suggest, the accuracy must be suspected of being fortuitous before it is accepted as very general.

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The Treatment of Boundary Terms in a Variational Principle Characterizing Transport Theory

In a recent note[1] Amster has presented a variational treatment of the diffusion theory boundary conditions that is purported to be more correct than a variational treatment by Pomraning and Clark (hereafter referred to as the PC treatment)[2]. His main argument for this position is the belief that his procedure is more straightforward and less arbitrary. In this note I would like to point out that the treatment of the diffusion theory (or higher order P_N) boundary conditions is, by necessity, arbitrary (even within the framework of the variational method), and further that Amster's treatment and the PC treatment are different in their aims.

Considering the first point, it is clear that the treatment of the P_N boundary conditions must be arbitrary. No finite expansion in full range polynomials can represent exactly the transport theory fluxes at the system boundaries since these fluxes are, in general, discontinuous in angle. The boundary conditions of Mark and Marshak (see Ref. 3 for a discussion) arise from completely different considerations and are a manifestation of this arbitrariness. In the PC treatment, which was an attempt to improve upon the boundary conditions of Mark and Marshak, the transport equation was characterized by a variational principle, including appropriate boundary terms in the functional. The use of a Legendre polynomial in angle trial function led to the usual P_N equations together with a set of nonlinear equations whose solution yielded a new set of boundary conditions applicable to all problems. Amster used the same functional to characterize the transport equation, but employed the classical Ritz method to determine the arbitrary constants in the general solution to a particular problem. Since this procedure leads to a set of linear equations with a unique solution, Amster argues that his procedure contains no arbitrariness and hence is basically more correct. There is an arbitrariness within the variational method, however, which Amster failed to consider and which strongly affects the result obtained by the Ritz method (as well as the result obtained by the PC treatment). This is an arbitrariness in the boundary terms of the functional itself. Consider a system with a free surface (no entering neutrons) at the left hand face, $z = a$. An appropriate functional is

$$F[\phi, \phi^*] = \int_a^\infty dz \int_{-1}^1 d\mu [\phi^* H\phi - S\phi^* - T\phi] +$$
$$+ \int_0^1 d\mu\, \mu \phi^*(a, \mu)\phi(a, \mu) +$$
$$+ \int_0^1 d\mu \alpha(\mu)\phi^2(a, \mu) +$$
$$+ \int_{-1}^0 d\mu \beta(\mu)\phi^{*2}(a, \phi) , \qquad (1)$$

where $\alpha(\mu)$ and $\beta(\mu)$ are *completely arbitrary* functions. The notation in Eq. (1) follows that in Ref. 2. The choice for these two functions will strongly affect the approximate solution found by the Ritz method. The main point to be emphasized is that the treatment of the P_N boundary conditions by the variational method is, unfortunately, arbitrary and it is difficult, if not impossible, to remove this arbitrariness.

The second main point to be made is that the results of Amster's treatment and the PC treatment are of a completely different nature. In the PC treatment the variational method is used to develop an approximate theory in the usual sense

[1] H. AMSTER, "A More Straightforward Use of Variational Principles with Boundary Conditions," *Nucl. Sci. Eng.*, this issue, p. 255.

[2] G. C. POMRANING and M. CLARK, Jr., *Nucl. Sci. Eng.*, **16**, 147-154 (1963).

[3] B. DAVISON and J. B. SYKES, *Neutron Transport Theory*, The Clarendon Press, Oxford, (1957).