to 38×10^5 cm^4/sec. They yield a corrected value of λ_t equal to 2.62 ± 0.04 cm at a graphite density of 1.60 g/gm^3, which can be compared with the value of 2.65 ± 0.03 cm obtained by pulsed-neutron methods with the same graphite. The effect of this correction upon the previously reported graphite absorption cross section of 3.44 ± 0.08 *mb*[1] is negligible since the correction is a minimum at long relaxation lengths.

It appears that the transport mean free path of thermal neutrons in graphite as measured by the poison method is in agreement with the value obtained by the pulsed-neutron method, after due account is made for diffusion-cooling effects.

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Degenerate Solutions to the Transport Equation With Anisotropic Scattering

Following the work of Case,[1] Mika[2] has shown that a complete set of eigenfunctions to the one-velocity transport equation for plane symmetry can be found when the scattering function may be expanded in a finite series of Legendre polynomials.

It may happen that some of the solutions to the homogeneous transport equation, corresponding to multiple order eigenvalues, are degenerate. In order to find the additional solutions necessary to complete the set of eigenfunctions, one has to take derivatives of the degenerate eigenfunction with respect to the eigenvalue (cf. Eq. B.5, p. 425, of Reference 2). In particular, when the eigenvalue ν_0 is a root of order two, one solution is $\psi_{\nu_0}(x,\mu)$, while according to Equation B.5, a second solution is

$$\psi_1(x,\mu) = \frac{\partial}{\partial\left(\frac{1}{\nu}\right)} \left[e^{-x/\nu} \quad \phi_\nu(\mu)\right]_{\nu=\nu_0}. \tag{1}$$

It is the purpose of this note to point out that this procedure is not always valid for anisotropic scattering and to indicate where the difficulty arises for a particular case. For a nonabsorbing medium ($c = 1$) there is always a double root at infinity, and the second solution corresponding to this root, as given by Equation (1), is correct for isotropic scattering, but incorrect for anisotropic scattering.

For simplicity, consider the case of linear anisotropic scattering. Let L be the linear transport operator for this case.

$$L = 1 + \mu\frac{\partial}{\partial x} - \frac{c}{2}\int_{-1}^{1} d\mu'(1 + s_1\,\mu\mu'), \tag{2}$$

where s_1 is the first-order scattering coefficient, which is equal to three times the mean cosine of the scattering angle in the laboratory system. The homogeneous transport equation with linear anisotropic scattering may then be written as

$$L\psi(x,\mu) = (1 + \mu\frac{\partial}{\partial x})\psi(x,\mu) - \frac{c}{2}\int_{-1}^{1} d\mu'(1+ s_1\,\mu\mu')\psi(x,\mu') = 0. \tag{3}$$

We seek solutions to Equation (3) of the form

$$\psi_\nu\,(x,\mu) = e^{-x/\nu} \quad \phi_\nu(\mu) \tag{4}$$

where, for the case $\nu \notin (-1,1)$, the discrete eigenfunctions are

$$\phi_\nu(\mu) = \frac{c\nu}{2(\nu-\mu)}\,[1 + (1-c)s_1\,\mu\nu]\,. \tag{5}$$

Operating on Equation (4) with L, one obtains

$$L\psi_\nu(x,\mu) = \frac{c}{2}\,e^{-x/\nu} \quad (1 + s_1\,\mu\nu)\,\Lambda(\nu) \tag{6}$$

where

$$\Lambda(\nu) = 1 - c\nu\tanh^{-1}\frac{1}{\nu} - s_1 c(1-c)\nu^2(\nu\tanh^{-1}\frac{1}{\nu} - 1). \tag{7}$$

Since $\Lambda(\nu) = 0$ is even in ν, the roots occur in pairs. Let $\xi = \frac{1}{\nu}$, and call the root having the smallest magnitude ξ_0. Since $\Lambda(\xi_0) = 0$, the solution corresponding to $\xi = \xi_0$ is the eigenfunction

$$\psi_0 = \frac{c}{2}\,e^{-x\xi_0}\,\frac{1}{1-\mu\xi_0}\left[1 + (1-c)\,\frac{s_1\mu}{\xi_0}\right]. \tag{8}$$

We note that for $|1 - c| << 1$, the root $\xi_0(c)$ to $\Lambda(\xi) = 0$ is given by $\frac{\xi_0^2}{(3-s_1)(1-c)} = 1 + 0(1-c)$.

Hence,

$$\lim_{c\to 1} \frac{\xi_0^2}{(3-s_1)(1-c)} = 1\,. \tag{9}$$

$\xi_0 = 0$ is therefore a double root for $c = 1$.

The eigenfunction corresponding to this root is given by Equation (8) in the limit as $c \to 1$: $\psi_0 = \frac{1}{2}$, since $\xi_0 \to 0$ as $c \to 1$ in the sense given by Equation (9).

[1] K. M. CASE, *Ann. Phys.* **9**, 1 (1960).
[2] J. R. MIKA, *Nucl. Sci. Eng.* **11**, 415 (1961).

As may be seen by direct substitution into Equation (3), the second discrete solution corresponding to $\xi_0 = 0$ is given by

$$\psi_1(x,\mu) = \frac{1}{2}\left[\frac{3}{3-s_1}\,\mu - x\right]. \tag{10}$$

But if one follows the procedure given by Equation (1),

$$\psi_1(x,\mu) = \lim_{c\to 1}\frac{\partial}{\partial\xi}\left[e^{-x\xi}\,\phi_\xi(\mu)\right]_{\xi_0}$$

$$= \frac{1}{2}\left[\left(\frac{3-2s_1}{3-s_1}\right)\mu - x\right], \tag{11}$$

which is not a valid solution.

The difficulty is revealed by a more careful examination of Equation (6), which we rewrite as

$$L\psi_\xi(x,\mu) = \frac{c}{2}\,e^{-x\xi}\,(1+\frac{s_1\mu}{\xi})\,\Lambda(\xi)\,. \tag{12}$$

It follows that

$$L\psi_0 = \frac{c}{2}\,e^{-x\xi_0}\left(1+\frac{s_1\mu}{\xi_0}\right)\Lambda(\xi_0) = 0$$

and $\psi_0 = \psi_{\xi_0}$ is indeed a solution to the transport Equation (3).

A second solution is usually obtained by writing

$$\frac{\partial}{\partial\xi}(L\psi_\xi) = L\left[\frac{\partial}{\partial\xi}\psi_\xi\right] = \frac{c}{2}\,e^{-x\xi}\left\{\left[-x\Lambda(\xi) + \frac{\partial}{\partial\xi}\Lambda(\xi)\right]\left(1+\frac{s_1\mu}{\xi}\right) - \Lambda(\xi)\,\frac{s_1\mu}{\xi^2}\right\} \tag{13}$$

and observing that when $c = 1$, not only is

$$\Lambda(0) = 0, \text{ but also } \left.\frac{\partial}{\partial\xi}\Lambda(\xi)\right|_{\xi=0} = 0.$$

But

$$\lim_{c\to 1} L\left[\left.\frac{\partial}{\partial\xi}\psi(\xi)\right|_{\xi_0}\right] = -\frac{1}{3}\,\mu s_1 \neq 0.$$

Thus, $\psi_1 = \left.\frac{\partial}{\partial\xi}\psi(\xi)\right|_{\xi=0}$ is a solution only if $s_1 = 0$, i.e., only for isotropic scattering in the laboratory system.

The appropriate modification of Equation (1) may be obtained by observing that for a constant A,

$$L(A\mu) = \left(\frac{3-s_1}{3}\right)A\mu\,.$$

Therefore,

$$\lim_{c\to 1} L\left[\left.\frac{\partial}{\partial\xi}\psi(\xi)\right|_{\xi_0} + \frac{3}{3-s_1}\left(\frac{1}{3}\,\mu s_1\right)\right]$$

$$= -\frac{1}{3}\,\mu s_1 + \frac{1}{3}\,\mu s_1 = 0.$$

Hence, the second solution for $\xi = 0$ is given by

$$\psi_1(x,\mu) = \left.\frac{\partial}{\partial\xi}\psi(\xi)\right|_{\xi=0} + \frac{\mu s_1}{3-s_1} \tag{14}$$

and not by Equations (1) or (11). With the additional term in Equation (14), the second solution becomes

$$\psi_1(x,\mu) = \frac{1}{2}\left[\frac{3}{3-s_1}\,\mu - x\right]$$

which agrees with the correct solution (10).

For higher-order scattering, the procedure becomes more complicated. Further terms have to be added to Equation (14) to provide valid second solutions to the transport equation.

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