where $\psi_s(\vec{r},\vec{\Omega})$ obeys the same spatial boundary conditions as Equations (1) and (2), and k_s represents the eigenvalue. Defining the appropriate adjoint boundary conditions, the adjoint density is given by

$$-v\vec{\Omega}\cdot\nabla\psi_j^+(\vec{r},\vec{\Omega}) + \sigma v\psi_j^+ - \int v\psi_j^+(\vec{r},\vec{\Omega}')\sigma(\vec{\Omega}\to\vec{\Omega}',\vec{r})d\vec{\Omega}'$$

$$= (1/4\pi)(v\sigma_f/k_j)\tilde{\psi}_j^+ , \qquad (5)$$

and hence from Equations (4) and (5) the orthogonality relation is given by

$$\left(\frac{1}{k_s} - \frac{1}{k_j}\right)\int v\sigma_f\tilde{\psi}_s\tilde{\psi}_j^+ d\vec{r} = 0.$$

Assuming the eigenfunctions to be normalized,

$$\int v\sigma_f\tilde{\psi}_s\tilde{\psi}_j^+ d\vec{r} = 4\pi\delta_{sj} , \qquad (6)$$

the time-dependent angular density, N, is expanded thusly:

$$N(\vec{r},\vec{\Omega};t) = \sum_s B_s(t)\psi_s(\vec{r},\vec{\Omega}). \qquad (7)$$

Using the above expansion in Equations (1) and (2) with the orthogonality as expressed by Equation (6) gives

$$(1/4\pi) p\sum_s \overline{B}_s(p)\,\ell_{sj} + [(1-k_j)/k_j]\overline{B}_j(p)$$

$$+\overline{B}_j(p)\sum_{i=1}^{m}\beta_i p/(p+\lambda_i) = \int\overline{S}(\vec{r},p)\tilde{\psi}_j^+ d\vec{r} \qquad (8)$$

where p represents the temporal Laplace transform variable, i.e.,

$$\overline{B}_s(p) = \int_0^\infty e^{-pt}B_s(t)dt, \qquad (9)$$

and

$$\ell_{sj} = \int\tilde{\psi}_s\tilde{\psi}_j^+ d\vec{r} . \qquad (10)$$

It has been tacitly assumed that the precursor concentration and angular density are initially zero. The above is to be compared with Cohen's[2] Equations (10.1) to (12). For a bare assembly assuming $v\sigma_f$ to be constant we can write

$$\ell_{sj} = 4\pi\delta_{sj}/(v\sigma_f) . \qquad (11)$$

Using Equation (11), the system of equations for $\overline{B}_j(p)$ becomes diagonal and hence

$$\overline{B}_j(p) = \left\{[v\sigma_f/(4\pi)]\tilde{\psi}_j^+(\vec{r}')\sum_{n=0}^{\infty} e^{-pn/R}\right\} /D_j, \qquad (12)$$

where

$$D_j = p + [(1-k_j)/k_j]v v\sigma_f + v v\sigma_f\sum_{i=1}^{m}\beta_i p/(p+\lambda_i). \qquad (13)$$

Now we cast the above, Equation (13), to the usual inhour equation by writing

$$v v\sigma_f = k_j/\ell_j \qquad (14)$$

where

$$\ell_j = k_j/(v v\sigma_f). \qquad (15)$$

Now clearly Equation (14) is modally independent and Equation (13) becomes the usual inhour equation,

$$D_j = p + \left[(1-k_j)/\ell_j\right] + (k_j/\ell_j)\sum_{i=1}^{m}\beta_i p/(p+\lambda_i). \qquad (16)$$

Using the above, the Laplace transform of the angular density is given by

$$\overline{N}(\vec{r},\vec{\Omega},p) \sim \sum_{s,n}\left\{\psi_s(\vec{r},\vec{\Omega})\tilde{\psi}_s^+(\vec{r}')e^{-pn/R}\right\} /D_s \qquad (17)$$

where the constant multiplier has been omitted. Equation (17) is identical to Equation (9) in the original $(k\beta/\ell)$ treatment[1] and hence the remaining development as given in Reference 1 follows as before.

Edward Garelis

General Electric Company
Vallecitos Atomic Laboratory

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The Effect of Chemical Binding on the Milne - Problem Extrapolation Distance

In a recent article on the thermal-neutron Milne problem[1] with isotropic scattering, expressions have been derived for the extrapolation distance Z_0 in the two extreme cases of very strong and zero energy exchange between neutrons and nuclei. These expressions are given below;

Strong Energy Exchange:

$$Z_0 = \frac{1}{\pi}\int_0^\infty dt$$

$$\left\{\frac{\frac{3}{t^2} - \frac{\frac{1}{\overline{\Sigma}}\int_0^\infty \frac{M(E)\Sigma_s(E)dE}{\Sigma_s^2(E) + t^2}}{\left[1 - \frac{1}{\Sigma_s t}\int_0^\infty M(E)\Sigma_s^2(E)\tan^{-1}\left(\frac{t}{\Sigma_s(E)}\right)dE\right]}\right\} \qquad (1)$$

[1] M. M. R. WILLIAMS, *Nucl. Sci. and Eng.* **18**, 260 (1964).

Zero Energy Exchange:

$$Z_0 = 0.7104 \ldots \frac{\overline{\ell_s^2}}{\overline{\ell_s}} \tag{2}$$

A bar indicates a Maxwellian average over the corresponding energy-dependent quantity.

For constant scattering cross section, both of these expressions become equal to the one-velocity value. Thus the more rapidly varying the cross section with energy, the greater the difference between (1) and (2) is likely to be. However, it is important to note that only the total cross section is involved and no other detailed properties of the scattering kernel are needed. Assuming that $\Sigma_s(E)$ can be represented by the asymptotic series derived by Wick[2] on the basis of the short-collision-time approximation, viz:

$$\Sigma_s(E) = \Sigma_f \left(1 + \frac{T_{\text{eff}}}{2AE} + \ldots\right) \tag{3}$$

we can insert this into (1) and (2) and expand in powers of $1/A$. Neglecting terms of order A^{-2} and higher, we find that in both cases;

$$Z_0 = 0.7104 \ldots \left(1 - \frac{T_{\text{eff}}}{2TA}\right) \ell_f + 0(A^{-2}) \tag{4}$$

Thus we have shown, to within the accuracy indicated, that the extrapolation distance for the non-absorbing Milne problem depends only on the energy variation of the total cross section.

The effect of chemical binding and thermal motion on Z_0 is illustrated clearly by (4). T_{eff} is a measure of the average kinetic energy of the scattering centre and contains some information on the dynamics of the scattering system. In all cases $T_{\text{eff}} \geq T$.

The effect of scattering anisotropy has not been included in the analysis of Reference 1; this can be equally important and is usually of opposite sign to the thermal effect. To a good first approximation, anistropy can be accounted for by replacing Σ_s by Σ_{tr} in (1) and (2); this amounts to adding a term $0.7104 \overline{\mu} \ell_f$ to (4), where $\overline{\mu}$ is the Maxwellian-averaged value of the mean cosine of the scattering angle in the laboratory system. The value of $\overline{\mu}(E)$ in the short-collision-time approximation is,

$$\overline{\mu}(E) = \frac{2}{3A}\left(1 - \frac{T_{\text{eff}}}{2AE}\right) + 0(A^{-3}). \tag{5}$$

The Maxwellian average is therefore,

$$\overline{\mu} = \frac{2}{3A}\left(1 - \frac{T_{\text{eff}}}{2AT}\right) + 0(A^{-3}) \tag{6}$$

or simply $2/3A$ since we are neglecting terms of $0\,(A^{-2})$.

Thus, finally, we can write for the extrapolation distance,

$$Z_0 = 0.7104 \ldots \left(1 - \frac{T_{\text{eff}}}{2TA} + \frac{2}{3A}\right) \ell_f. \tag{7}$$

If $T_{\text{eff}}/T = 4/3$, Z_0 assumes its one-velocity value of 0.7104 ℓ_f. For graphite, Parks[3] obtains $T_{\text{eff}} =$ 2.363 at 300 K and 1.432 at 600 K, this causes a decrease in Z_0, below its one-velocity value, of 9% and 4% respectively. It appears that the extrapolation distance is noticably temperature dependent; furthermore, the effect will be much greater at low temperatures where T_{eff}/T increases considerably. These results suggest that measurements of Z_0 at various temperatures would provide some useful information on the solid or liquid state structure of moderators or other materials.

An early experiment[4] on graphite gave $Z_0 = 1.69$ to 1.72 cm. The above expression then predicts that $T_{\text{eff}}/T = 3.2 \pm 0.5$, the experiments were presumably at room temperature although this is not stated. This result is in reasonable agreement with the Park's value.

Finally it should be mentioned that these results apply only to non-absorbing media. For finite absorption, the asymptotic spectrum is no longer a Maxwellian in equilibrium with the moderator and the extrapolation distance will then be more sensitive to the scattering kernel. By analogy with the one-velocity case we can expect absorption to introduce corrections to Z_0 of the order of $1-C = \overline{\Sigma_a}/\overline{\Sigma_s}$; this is very much smaller than the thermal motion and anisotropy effects for the common moderators and can be neglected. However, (1) and (2) can be generalized to include absorption if necessary but the amount of numerical work involved is large.

M. M. R. Williams

Physics Department
Birmingham University
England

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