# Letters to the Editor

## Eigenvalues of the Neutron Transport Equation with Anisotropic Scattering

In a recent paper,[1] Dawn and Chen have discussed the existence of discrete eigenvalues of the one-group neutron transport equation with anisotropic scattering. Their treatment refers to eigenvalues $\nu$ for a spatial dependence of $\exp(-x/\nu)$ in a medium with a stationary field and characterized by $c$, the number of secondary neutrons per collision. Such a system is equivalent to a time-dependent one, where the spatial variation is described by the buckling approximation. Apparently, the authors have not been aware of the fact that such time-dependent systems have been treated in several works[2-8] and that detailed numerical results are available.

A comparison of the work by Dawn and Chen[1] with the works of the present author[4-8] can easily be done by noting that the spatial dependences in the two cases are related by

$$Bl_s = \frac{i}{\nu c} , \tag{1}$$

where $l_s$ is the mean-free-path for scattering and $B$ (or $B^2$) denotes the buckling. The generalized decay constant is related to $c$ by

$$\Lambda = 1 - \frac{1}{c} . \tag{2}$$

The Legendre coefficients $f_n$ used by Dawn and Chen[1] are identical to our $b_n$.

Let us first discuss the case of linear anisotropy with $0 \leqslant b_1 = f_1 \leqslant \frac{1}{3}$. The curves of the decay constant versus buckling in Fig. 1 may be compared to the right-hand part of Fig. 3 in Ref. 1. It should be noted that the eigenvalues in $Bl_s$ or $\nu$ always occur in pairs with plus and minus signs. This means that two such eigenvalues will coincide on curves plotted as in Fig. 1. The complete equivalence between the results obtained in Refs. 4 and 1 of the eigenvalues $Bl_s$ and $\nu$ is seen from Table I. The region $c > (1 + 1/b_1)$ is not discussed by Dawn and Chen.[1] As can be seen in Fig. 1, there are two pairs of real eigenvalues here, but if $b_1 > \pi^2/48 = 0.2056$ they may go over into complex eigenvalues instead. The limit $b_1 = \pi^2/48$ and the existence of complex eigenvalues already was derived by Davison.[9] Numerical values of the real[4] and complex[6] eigenvalues in this region have been given in great detail. It is therefore quite clear that region II in Fig. 3 of Ref. 1 contains four eigenvalues, either purely imaginary or complex.

In a similar way Fig. 4 of Ref. 1 may be compared to Fig. 2 in Ref. 7, and it will be seen that the eigenvalues calculated in Ref. 7 are in accordance with the predictions in Ref. 1.

[1]T. DAWN and I. CHEN, *Nucl. Sci. Eng.*, **72**, 237 (1979).

[2]I. KUŠČER, *Nucl. Sci. Eng.*, **38**, 175 (1969).

[3]I. CAPRINI and V. PROTOPOPESCU, *Rev. Roum. Phys.*, **17**, 637 (1972).

[4]N. G. SJÖSTRAND, "Decay Constants for Pulsed Monoenergetic Neutron Systems with Anisotropic Scattering," CTH-RF-28, Chalmers University of Technology (1975).

[5]N. G. SJÖSTRAND, *J. Nucl. Sci. Technol.*, **12**, 256 (1975).

[6]N. G. SJÖSTRAND, *J. Nucl. Sci. Technol.*, **13**, 81 (1976).

[7]N. G. SJÖSTRAND, "Decay Constants for Pulsed Monoenergetic Neutron Systems with Quadratically Anisotropic Scattering," CTH-RF-30, Chalmers University of Technology (1977).

[8]N. G. SJÖSTRAND, *Atomkernenergie*, **31**, 16 (1978).

[9]B. DAVISON, "Milne Problem in a Multiplying Medium with a Linearly Anisotropic Scattering," CRT-358, National Research Council of Canada (1946).

TABLE I

Comparison of Eigenvalues in the Stationary Case (Ref. 1, Fig. 3) and Time-Dependent Case (Fig. 1)

| Stationary Case | | Time-Dependent Case | |
|---|---|---|---|
| Region | Types of Eigenvalues | Region | Types of Eigenvalues |
| $0 < c < 1$ | 2 real | $\Lambda < 0$ | 2 imaginary |
| $1 < c < (1 + 1/3b_1)$ | 2 imaginary | $0 < \Lambda < \left(\frac{1}{1 + 3b_1}\right)$ | 2 real |
| $(1 + 1/3b_1) < c < (1 + 1/b_1)$ | 2 real<br>2 imaginary | $\left(\frac{1}{1 + 3b_1}\right) < \Lambda < (1 - b_1)$ | 2 real<br>2 imaginary |
| $(1 + 1/b_1) < c$ | 4 imaginary or<br>4 complex | $(1 - b_1) < \Lambda$ | 4 real or<br>4 complex |

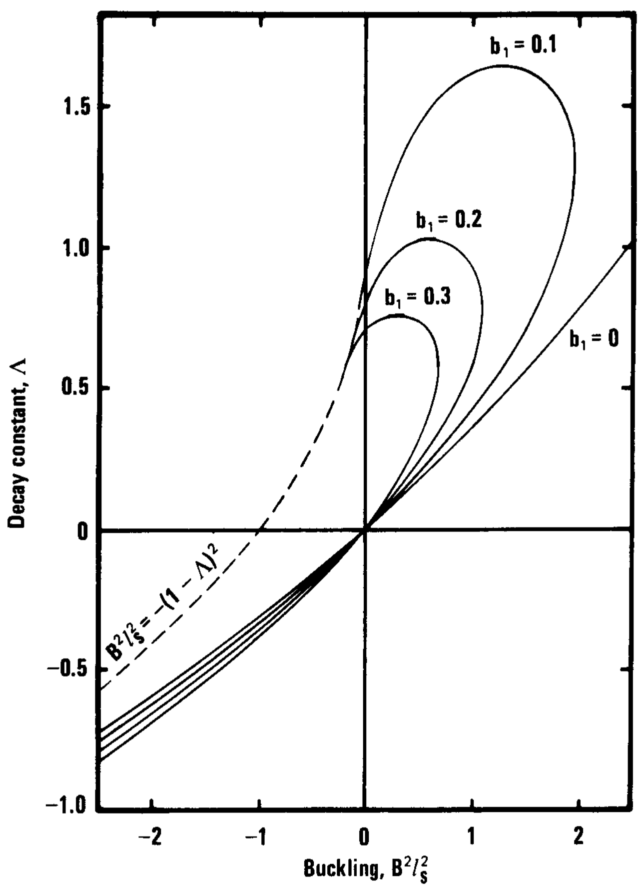


Fig. 1. The generalized decay constant as a function of buckling for various degrees of linearly anisotropic scattering.

Also the more general Figs. 1 and 2 in Ref. 1 agree with the results obtained in Refs. 7 and 8 (apart from the fact that the vertical border lines in Fig. 1 should have been drawn at $1/f_1$ at $\pm 3$ instead of at $\pm 4$). From Ref. 7 it is clear that region II in Fig. 2 of Ref. 1 frequently contains four imaginary eigenvalues. When they disappear, it is probable that they go over into four complex eigenvalues, just as is the case for purely linear anisotropy. However, this has not been confirmed by calculations.

*N. G. Sjöstrand*

Department of Reactor Physics
Chalmers University of Technology
S-41296 Göteborg, Sweden

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## Reply to a Comment on "Eigenvalues of the Neutron Transport Equation with Anisotropic Scattering"

The original idea in our paper[1] was to give a general analytical method to determine the number and the mathematical property (real, purely imaginary, or complex) of the discrete eigenvalues of the monoenergetic neutron transport equation with anisotropic scattering. Sjöstrand's comment[2] suggested that his numerical study on the time-dependent transport problem can be compared with our work and his numerical result show agreement with our analytical prediction.

Sjöstrand[2] also discussed the existence of the complex discrete eigenvalues. Just as illustrated in Sec. IV of Ref. 1, the difficulty is how to determine the condition on parameters $c, f_1, f_2, \ldots$ for the existence of complex eigenvalues, where $c$ is the number of the secondary neutrons per collision and $f_1, f_2, \ldots$ are the Legendre coefficients of the scattering function. This problem for the linearly anisotropic scattering case had been considered by Thielheim and Claussen.[3] In Fig. 1 of their paper, they give a boundary of the complex discrete eigenvalues without showing how to determine such a boundary. A very basic property of the quadratically algebraic equation

$$x^2 + 2ax + b = 0 \ , \tag{1}$$

may help us to solve this problem. This property is that the boundary of the complex roots of Eq. (1), $a^2 - b = 0$, is the condition that Eq. (1) has double roots. With this in mind, we may ask whether this simple property is true for the present problem. That this property is correct for the linearly anisotropic scattering case can be shown as follows.

The discrete eigenvalues for the linearly anisotropic scattering case are roots of the equation

$$\begin{aligned} \Lambda(\nu^2) &= 1 + 3cf_1(1-c)\nu^2 - c[1 + 3f_1(1-c)\nu^2]f(\nu^2) \ , \\ &= 0 \ , \end{aligned} \tag{2}$$

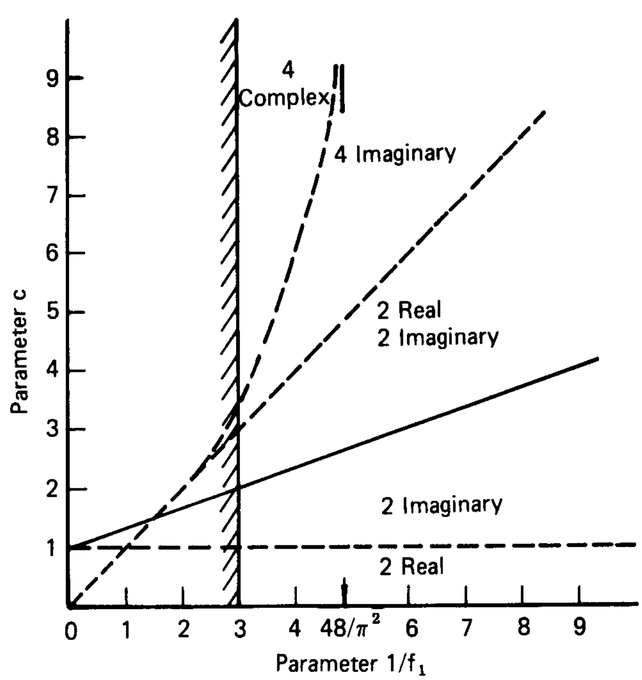


Fig. 1. Classification of discrete eigenvalues in parameter space for linearly anisotropic scattering.

[1] T. DAWN and I. CHEN, *Nucl. Sci. Eng.*, 72, 237 (1979).

[2] N. G. SJÖSTRAND, *Nucl. Sci. Eng.*, 74, 154 (1980).

[3] K. O. THIELHEIM and K. CLAUSSEN, *Kernenergie*, 16, 321 (1973).