$$\lambda(\mu)\psi(x,\mu) - P\int_{-1}^{1} \frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)}{\nu-\mu}\,d\nu =$$

$$e^{-\frac{b+x}{\mu}}\left[\lambda(\mu)\psi(-b,\mu) - P\int_{-1}^{1} \frac{\left(\frac{c}{2}\right)\nu\psi(-b,\nu)}{\nu-\mu}d\nu\right];\ \mu>0, \quad (10)$$

where

$$\lambda(\mu) = \frac{1}{2}\left[\Lambda^{+}(\mu) + \Lambda^{-}(\mu)\right] = 1 - c\mu\tanh^{-1}\mu, \quad (10a)$$

and $\Lambda^{\pm}(\mu)$ are the boundary values of the function

$$\Lambda(z) = 1 - \frac{c}{2}z\int_{-1}^{1}\frac{d\nu}{z-\nu} = 1 - cz\tanh^{-1}\frac{1}{z} \quad (11)$$

as it approaches the cut $[-1,1]$ from above and below the real axis, respectively.

If the boundary condition requires that the incoming distribution is zero:

$$\psi(-b,\mu>0) = \psi(b,\mu<0) = 0\,, \quad (12)$$

Equation (10) reduced to

$$\lambda(\mu)\psi(x,\mu) - P\int_{-1}^{1} \frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)}{\nu-\mu}\,d\nu$$

$$= -e^{-x/\mu}f(b,\mu);\ \mu>0, \quad (13)$$

where we have defined

$$f(b,\mu) = e^{-b/\mu}\int_{0}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(b,\nu)}{\nu+\mu}\,d\nu\,. \quad (14)$$

The singular equation for $\psi(x,\mu<0)$ also follows from the foregoing results by symmetry.

Now consider

$$\psi(x,z) = \frac{c}{2}\frac{1}{z}\int_{-b}^{x}\rho(y)\,e^{-\frac{x-y}{z}}dy;\quad Re(z)>0, \quad (15)$$

which is recognized as the extension of (2), subject to (12), to the right half of the complex plane. The analysis leading to (10) gives, in this case,

$$\Lambda(z)\psi(x,z) = \int_{-1}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)d\nu}{\nu-z} - e^{-x/z}f(b,z). \quad (16)$$

Since the left side of (16) vanishes at the roots $\pm\nu_0$ of $\Lambda(z) = 0$, the right side must also vanish at these two points. Hence we have the two conditions

$$\int_{-1}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)}{\nu\mp\nu_0}d\nu = e^{\mp x/\nu_0}f(b,\pm\nu_0). \quad (17)$$

Setting $x = b$ in (13) and (17), we get

$$\lambda(\mu)\psi(b,\mu) - P\int_{0}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(b,\nu)}{\nu-\mu}d\nu = -e^{b/\mu}f(b,\mu), \quad (18)$$

and

$$f(b,\nu_0) = f(b,-\nu_0), \quad (19)$$

respectively. These last two equations for the emerging distribution correspond, apart from notation, to the results derived, in a somewhat different way, by Leonard and Mullikin[2].

The advantage of this transformation arises from the fact that a general method for treating singular integral equations is available[3]. Consequently, (10) is a simpler starting point than the original transport equation (1). Moreover, this transformation provides a different way of treating transport problems in this geometry from the method of singular expansion modes developed by Case[4]. What is more important, the success of this method in plane geometry serves as the prime motivation for investigating the transform properties of the integral equations in spherical and cylindrical geometries where the eigenfunction expansions become intractable[1]. The application of this approach to the one-dimensional critical problem, as well as a comparison with the results obtained previously by means of Case's formulation[5], is given in Reference 1.

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Effective Surface in Lattices in the Calculation of Resonance Integrals

The exact and approximate method for the calculation of the Dancoff Factor (C) has been examined by many people[1-3]. However, little attention has yet been paid to the relation between the effective surface (S_{eff}) and the Dancoff Factor.

Recently, Levine[4,5] obtained a new relation empirically from a Monte Carlo study and has discussed it in terms of the Bell approximation for the collision probability. His expression is

[1] S.M. DANCOFF and M. GINSBURG, "Surface Resonance Absorption in a Close Packed Lattice," CP-2157, (1944).
[2] V. NAMIAS, "Calculation of Dancoff Correction," WCAP-1097, (1959).
[3] YUZO FUKAI, *Nucl. Sci. Eng.* 9, 370, (1961).
[4] M. M. LEVINE, *Trans. Am. Nucl. Soc.* 5, No. 2, 373, (1962).
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$$D_{eff} = S_{eff}/S = (1-C)/(1+0.1C) \quad (1)$$

where S is the actual fuel-surface area. This relation is discussed below in connection with a study of the escape probability (P) in lattices.

We have examined the Bell approximation for the escape probability both for square and slab lattices. The Bell approximation for the escape probability from the absorber lump in lattices (P_B) is[6],

$$P_B = (1-C)P_0/[1-C(1-A)] \quad (2)$$

$$A = \Sigma_0 \ell_0 P_0 \quad (3)$$

where P_0 is the escape probability for an isolated fuel rod,

Σ is the macroscopic total cross section

and ℓ is the mean chord length.

Suffixes 0 and 1 denote the absorber and moderator region. The main approximation in (2) is that the collision probability of incoming neutrons in the absorber lump is replaced by the one for an isolated lump, A. Since the collision probability depends to some extent on the angular distribution of the incoming neutrons, (2) is only exact for the absorber lumps distributed randomly and may fail for the systematic lattice. For hexagonal lattices, (2) was examined at BNL by Monte Carlo calculations and shown to be satisfactory[7]. The accuracy of the approximation, however, depends on the lattice arrangement, so (2) has been checked below for square and slab lattices.

KITTY HAWK[8] was used for one-velocity Monte Carlo calculation in square lattices. Σ_0, Σ_1 and d (distance between rods) are taken as parameters, which are shown in Table I. The radius of the absorber rod was fixed to 0.40 cm because one of four parameters is only a scaling factor. P and P_B are shown in Table I. The C factor in the calculation of P_B is obtained taking 16 nearest neighbors into consideration with the partial shadowing[3]. As is seen from Table I, P is well approximated by P_B with the error less than 1%. If we define C_0 by (2), in which P_B is replaced by P, C_0 can be taken as constant and equal to C, for the range of $\Sigma_0 \ell_0 > 0.4$, with the uncertainty of 3%.

For slab geometry, P can be obtained by an infinite series of E_3 functions[9].

$$P = \frac{1}{b_0}\int_1^\infty \frac{(1-e^{-b_0 x})(1-e^{-b_1 x})}{x^3[1-e^{-(b_0+b_1)x}]}\,dx$$

$$= \frac{1}{2b_0}[1-2E_3(b_0) - 2E_3(b_1) + 2F] \quad (4)$$

$$F = \sum_{m=1}^{\infty}\{2E_3[m(b_0+b_1)] - E_3[b_0+m(b_0+b_1)] - E_3[b_1+m(b_0+b_1)]\} \quad (5)$$

where b_i is the multiple of the thickness by Σ_i,

and $E_3(x) = \int_1^\infty e^{-xu}\,u^{-3}\,du$.

The convergence of the series is very fast. P and P_B for slab lattices are shown in Table II. C for slab is given by $2E_3(b_1)$. The error of P_B is within 5% of P. The error is, however, about 30% of the change of P due to the lump interaction in the above range. Since this change is important, the approximation (2) in slab geometry is not satisfactory for transparent absorbers.

TABLE I

Escape Probabilities for Square Lattices
(radius of cylindrical absorber = 0.40 cm)

Σ_1(cm^{-1})	d(cm)	$2\Sigma_0 R_0$	P	P_B
1.0	1.0983	0.4	0.64760 ± 0.00228	0.64593
	(1.4)[a]	0.8	0.47565 ± 0.00302	0.47417
		2.0	0.25335 ± 0.00135	0.25694
	1.2198	0.4	0.69610 ± 0.00219	0.68821
	(1.96)[a]	0.8	0.52165 ± 0.00414	0.52116
		2.0	0.29420 ± 0.00181	0.29270
1.4	1.0983	0.4	0.69330 ± 0.00220	0.68878
		0.8	0.52469 ± 0.00414	0.52184
		2.0	0.29353 ± 0.00181	0.29323
	1.2198	0.4	0.72825 ± 0.00212	0.72286
		0.8	0.56252 ± 0.00573	0.56198
		2.0	0.32703 ± 0.00239	0.32593

[a]Moderator to absorber volume ratio

TABLE II

Escape Probabilities for Slab Lattices

$2b_1$	$2b_0$	P	P_B
0.4	0.6	0.49297	0.46809
(0.51457)[a]	1.0	0.36721	0.35013
	2.4	0.18877	0.18422
0.6	0.6	0.55785	0.53392
(0.38310)[a]	1.0	0.43174	0.41373
	2.4	0.23409	0.22859
0.8	0.6	0.59439	0.57347
(0.28865)[a]	1.0	0.47086	0.45418
	2.4	0.26481	0.25920
1.0	0.6	0.61674	0.59919
(0.21938)[a]	1.0	0.49604	0.48145
	2.4	0.28626	0.28101

[a]C Factor

[6]G. I. BELL, *Nucl. Sci. Eng.* **5**, 138 (1960).

[7]L. W. NORDHEIM, *Symp. Appl. Math.*, **11**, 58 (1961).

[8]R. A. DANNELS and S. M. HENDLEY, "KITTY HAWK – A Monoenergetic Monte Carlo Code to Calculate Neutron Capture Probabilities," WCAP-2256 (1963).

[9]W. ROTHENSTEIN, "Collision Probabilities and Resonance Integrals for Lattices," BNL-563 (T-151) (1959).

Now let us denote the effective surface for *monoenergetic* neutrons by S_e. S_e is obtained by,

$$P(\Sigma_0 \ell_0, C) = P_0(\Sigma_0 \ell_{0e}) \quad (6)$$

$$S_e/S = \ell_0/\ell_{0e} = D_e. \quad (7)$$

For a square lattice of cylindrical absorbers, P is approximated by P_B. Expressions for P_0 in the limit of large and small $\Sigma_0 \ell_0$ can be derived from the series expansion of Case *et al*[10] as follows:

$$P_0 = 1 - 2/3\ \Sigma_0 \ell_0 \quad (\Sigma_0 \ell_0 \ll 1)$$

and

$$P_0 = \frac{1}{\Sigma_0 \ell_0} \quad (\Sigma_0 \ell_0 \gg 1)$$

so that S_e/S approaches $(1-C)$ when $\Sigma_0 \ell_0$ becomes infinite and approximately $(1-C)/(1+C/2)$ when $\Sigma_0 \ell_0$ becomes zero. S_e/S is shown in Figure 1 as a function of $\Sigma_0 \ell_0$ in intermediate range. S_e/S becomes below 90% of $(1-C)$ for the range of $\Sigma_0 \ell_0 \leq 1$.

Similar curves can be obtained for slab geometry if the approximation (2) is used. It is not correct, however, to apply this to the slab geometry, and S_e/S for slab is obtained using the exact P (shown in Figure 1). S_e/S in slab geometry is a complicated function of b_0 and b_1. However, the deviation from the value $(1-C)$ is small and the average of S_e/S with a weighting function can be taken $(1-C)$ with a little error.

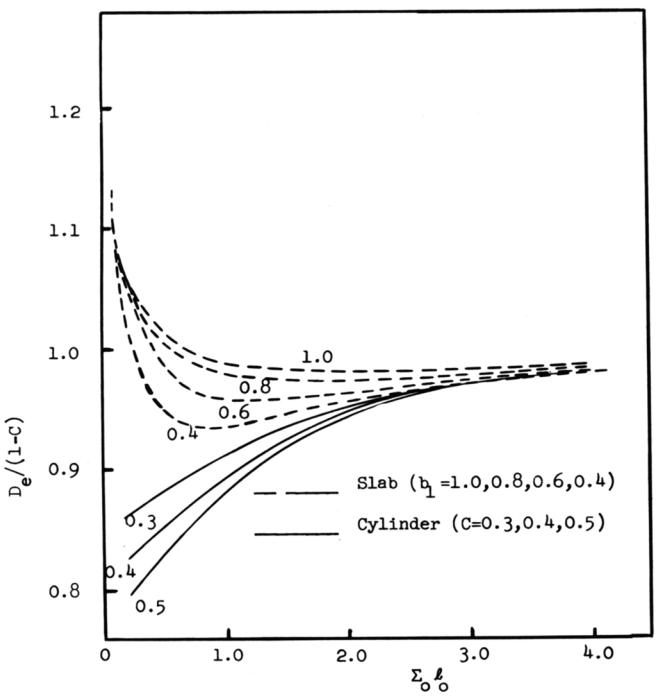

Fig. 1. Effective surface for monoenergy neutrons ($De = Se/S$) as a function of $\Sigma_0 \ell_0$.

For square and hexagonal lattices, it is necessary to perform an integration to obtain S_{eff}/S. For a special case, it is possible by an analytical approach. If the potential scattering in the absorber lump is zero and the *NR* approximation is adopted, D_{eff} is given by

$$D_{eff} = \ell_0/\ell_{0eff}. \quad (8)$$

Further, $$\int \sigma_a P_0(\Sigma_0 \ell_{0eff})du = \int \sigma_a P(\Sigma_0 \ell_0, C)du \quad (9)$$

where σ_a is the microscopic absorption cross section of the fuel element. The integration can be performed for the non-Doppler broadened case, if the corrected Wigner's rational approximation[9] is used for P_0.

$$P_0 = \frac{1}{1 + \Sigma_0 \ell_0 - \dfrac{h \Sigma_0 \ell_0}{1 + h \Sigma_0 \ell_0}}. \quad (10)$$

After the integration and arrangements,

$$\int \sigma_a P(\Sigma_0 \ell_0, C)du = \frac{\pi}{2} \frac{\Gamma_\gamma}{E_0} \sqrt{\frac{\sigma_0 f(C)}{N \ell_0}} \quad (11)$$

$$f(C) = (1-C)\ \frac{1 + h(1-C) + 2\sqrt{h(1-C)}}{1 + 2\sqrt{h(1-C)}} \quad (12)$$

where E_0 is the energy,
Γ_γ is the absorption width of the resonance
σ_0 is the resonance peak cross section and
N is the atomic number density of the absorber.

The effective surface is obtained from these by

$$D_{eff} = f(C)/f(o). \quad (13)$$

To represent the second term correctly in the limit of $\Sigma_0 \ell_0 << 1$, h should be taken as 1/3. The dependence of D_{eff} on C is shown in this case in Figure 2. This is well approximated by (1) for $C = 0 \sim 0.5$.

D_{eff} for the more general case is obtained by the ZUT code[11], in which lattice interaction is taken into consideration by (2). The actual values were taken for the physical constants of UO_2. The radius of the absorber lump and C factor were changed arbitrarily. At first D_{eff} for separate resonance levels were examined. The results show that the correction factor to $(1-C)$ for D_{eff} is about 13% and 7% of C for the levels of 6.68 eV and

[10] K. M. CASE, F. de HOFFMAN and G. PLACZEK, *Introduction to the Theory of Neutron Diffusion*, Government Printing Office, (1953).

[11] G. F. KUNCIR, "ZUT, TUZ: A Program for the Calculation of Resonance Integrals," GA-2525, (1961).

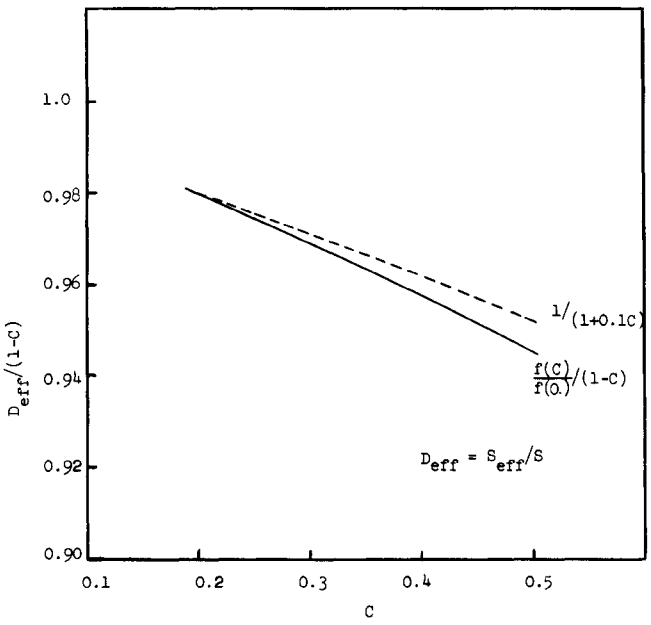

Fig. 2. Effective surface for a resonance without Doppler broadening (cylindrical lump).

36.8 eV, respectively, at the temperature of zero K. At 293 K, these become about 12% and 7%, respectively. The dependence of the correction factor on ℓ_0 is not distinct for the range of $\Sigma_{p0}\ell_0 = 0.16 \sim 0.64$, where Σ_{p0} is the macroscopic potential scattering cross section of UO_2, although a slight increase is seen in the correction factor with the increase of ℓ_0 which is expected from Figure 1.

Calculations for all resolved levels were run at 293 K to obtain the relation between D_{eff} and C in total resonance absorption in U^{238}. The results again show that (2) is a good approximation. The ten per cent correction of C for D_{eff} is considered as the average for all resonance levels.

As a conclusion, Levine's relation, (1), is well suited to the calculation of D_{eff} for square and hexagonal lattices for the normal absorber radius although it is questionable to extend this to an unpractically large absorber, say over 1 in $\Sigma_{p0}\ell_0$, as can be inferred from the curve in Figure 1. It is recommended, however, to use the classical factor $(1-C)$ for D_{eff} in slab geometry.

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Fast Fission Factors in Slightly Enriched Uranium, Light-Water-Moderated Slab Lattices*

The fast fission factor of reactor core assemblies is usually inferred from measurements of the ratio of U^{238} to U^{235} fission rates in the fuel of the lattice ($= \delta_{28}$). These measurements are generally normalized to the results of an auxiliary experiment, the so-called double-fission-chamber experiment[1], whose uncertainty heavily contributes to the overall margin of error of the data.

The present note describes a measurement of δ_{28} in slightly enriched uranium/light-water-moderated slab lattices, performed by a technique which differs to some extent from the methods currently used[1,2,3] and yields data of satisfactory accuracy.

The lattice investigated consisted of an array of bare U metal slabs enriched to 1.25% U^{235}. The slabs were arranged in rows of continuous sheets, 0.122 in. thick, spaced to give the desired volume ratios. The water-to-uranium volume ratios investigated were nominally 4:1, 3:1, 2:1, 1.5:1, and 1:1.

The facility used was a miniature assembly, measuring 16 in. high by 12 in. wide by 12 in. long, contained in an aluminum tank surrounded on all sides, except the top, by $\frac{1}{16}$ in. of cadmium sheet and 3 in. of paraffin. The whole assembly was irradiated in the tunnel under the Brookhaven National Laboratory graphite reactor. Previous work had proved the feasibility of such small-size assemblies for measurements of microscopic lattice parameters[1].

The ratio of U^{238} to U^{235} fission rates in the uranium was deduced from the fission-product activities of two uranium foils, one enriched to 1.25% U^{235} and one depleted to about 3 parts/10^6 U^{235}, irradiated bare at equal flux positions in the fuel of the lattice. The foils were rectangular (0.005 in. thick by 0.122 in. wide by 0.500 in. long) and were inserted into the slab sandwiched between 0.001 in. Al catchers, as shown in Figure 1. The irradiation time was 40 min.

The induced gamma activities of the detectors were measured by a 2 in. dia. × 2 in. high NaI(Tl) single-channel analyzer. The beta background was eliminated by a $\frac{1}{8}$ in.-thick pure Al shield. Discrimination against the activity from the U^{238} capture products (U^{239} and Np^{239}) was achieved by

*Work performed under the auspices of the USAEC.

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