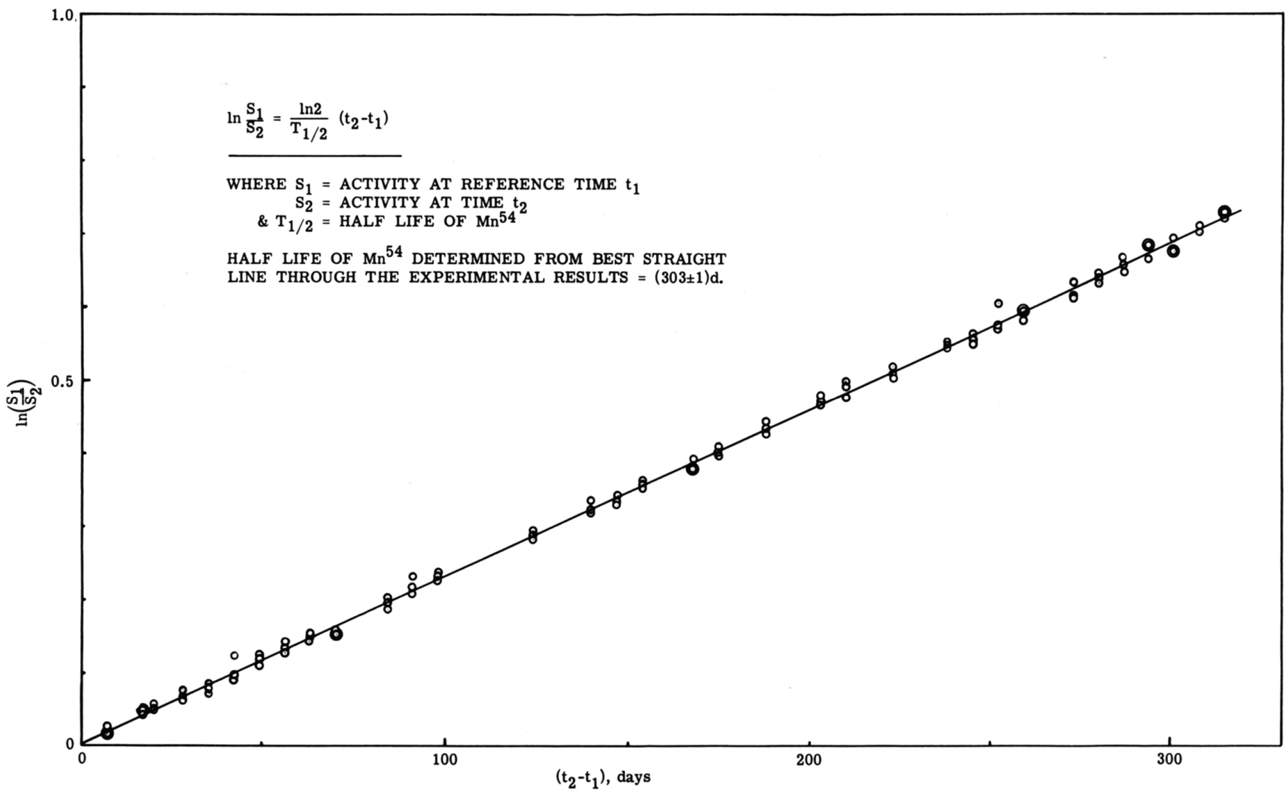

Fig. 1. Radioactive decay of Mn^{54} present in irradiated foils of iron enriched in Fe^{54}.

λ_{54} = decay constant of Mn^{54}

f_{58} = T.P.A. ion chamber factor for Fe^{59}

f_{54} = T.P.A. ion chamber factor for Mn^{54}

ϕ_{th} = thermal neutron flux

ϕ_f = fast neutron flux

t_i = irradiation time

and t_0 = time between end of irradiation and measurement.

This equation is only approximate in that it neglects the activity of Co^{60} produced subsequent to the decay of Fe^{59}, but this is unimportant when the ratio of thermal- to fast-neutron flux is less than about 5. Enriched iron foils have been irradiated together with nickel and cobalt monitors for 41 d in a monitoring stringer in the graphite moderator of the Advanced Gas-Cooled Reactor at Windscale, where the ratio of thermal- to fast-neutron flux is about 20. In this experiment C can be determined more accurately than for irradiations in hollow fuel element facilities in materials testing reactors where the ratio of thermal- to fast-neutron flux is at least 10 times lower. The value of C obtained from the AGR irradiation, allowing for the additional activity of Co^{60}, which is ~20% of the activity of Fe^{59} in this case, is $(4.0 \pm 0.8) \times 10^{-3}$. The values of a_{58} and f_{58} are not known with sufficient accuracy to enable this measured value of C to be applied to irradiations of foils with different a_{58}/a_{54} ratios.

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Received March 2, 1964

On the Transport Equation in Plane Geometry*

We propose to show that the stationary, monoenergetic transport equation, under the restrictions of plane symmetry and isotropic scattering, is equivalent to a singular integral equation with the space variable appearing as a parameter. This transformation was suggested by the work of Leonard and Mullikin[2], where complex transforms of

*This work is part of the doctoral thesis[1] submitted to the Nuclear Engineering Department of the University of Michigan and performed under the auspices of the U.S. Atomic Energy Commission.

the plane and spherical neutron densities are used in conjunction with the respective integral equations for the densities to arrive at exact criticality expressions for the two geometries.

Under the above assumptions, the angular neutron density ψ depends on the single-position variable $x(-b \leq x \leq b)$ and the direction variable $\Omega_x = \mu$, and satisfies the equation

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')d\mu' , \quad (1)$$

where c is the mean number of secondaries per collision, and distance is measured in units of the total mean free path.

Alternatively, (1) can be written in the integral equation form:

$$\psi(x,\mu) = e^{-\frac{b+x}{\mu}} \psi(-b,\mu) + \frac{c}{2}\frac{1}{\mu} \int_{-b}^{x} \rho(y) e^{-\frac{x-y}{\mu}} dy, \ \mu > 0, \quad (2)$$

where $\psi(-b,\mu > 0)$ is the incoming distribution at the boundary, and $\rho(x)$ denotes the neutron density:

$$\rho(x) = \int_{-1}^{1} \psi(x, \nu)d\nu = \int_{0}^{1} [\psi(x,\nu) + \psi(x,-\nu)] \ d\nu. \quad (3)$$

The expression for ψ for negative μ can be determined by replacing x by $-x$ in (2) and making use of the symmetry conditions

$$\psi(-x,\mu) = \psi(x,-\mu); \quad \rho(-y) = \rho(y): \quad (4)$$

$$\psi(x,-\mu) = e^{-\frac{b-x}{\mu}} \psi(b,-\mu) + \frac{c}{2}\frac{1}{\mu} \int_{x}^{b} \rho(y) e^{-\frac{y-x}{\mu}} dy, \ \mu > 0 . \quad (5)$$

The proposed singular integral equation for $\psi(x,\mu)$ is now obtained by using (3) to eliminate $\rho(y)$ from (2). Substitution of (3) into (2) and change of the order of y and ν integration gives

$$\psi(x,\mu) = e^{-\frac{b+x}{\mu}} \psi(-b,\mu) + \frac{c}{2}\frac{1}{\mu} \int_{0}^{1} d\nu \int_{-b}^{x} [\psi(y,\nu) + \psi(y,-\nu)] \ e^{-\frac{x-y}{\mu}} dy, \ \mu > 0. \quad (6)$$

The integral over y in (6) can be performed by inserting the explicit forms for $\psi(x,\pm\nu)$ from (2) and (5). Thus, for example, we have

$$\int_{-b}^{x} \psi(y,\nu) e^{-\frac{x-y}{\mu}} dy = \psi(-b,\nu) \int_{-b}^{x} e^{-\frac{b+y}{\nu}} e^{-\frac{x-y}{\mu}} dy + \frac{c}{2}\frac{1}{\nu} \int_{-b}^{x} e^{-\frac{x-y}{\mu}} dy \int_{-b}^{y} \rho(z) \ e^{-\frac{y-z}{\nu}} dz. \quad (7)$$

The first term on the right in (7) yields

$$\frac{\nu\mu}{\nu-\mu} \ \psi(-b,\nu) \left[e^{-\frac{b+x}{\nu}} - e^{-\frac{b+x}{\mu}} \right] . \quad (7a)$$

When we interchange the order of y and z integrations in the second term, we get

$$\frac{c}{2}\frac{1}{\nu} \int_{-b}^{x} e^{-\frac{x-y}{\mu}} dy \int_{-b}^{y} \rho(z) \ e^{-\frac{y-z}{\nu}} dz$$

$$= \frac{c}{2}\frac{1}{\nu} \int_{-b}^{x} \rho(z) \ dz \int_{z}^{x} e^{-\frac{x-y}{\mu}} e^{-\frac{y-z}{\nu}} dy$$

$$= \frac{\mu}{\nu-\mu} \left[\nu \psi(x,\nu) - \mu\psi(x,\mu) \right] - \frac{\mu}{\nu-\mu} \left[\nu\psi(-b,\nu) e^{-\frac{b+x}{\nu}} - \mu \psi(-b,\mu) e^{-\frac{b+x}{\mu}} \right] , \quad (7b)$$

where we have made use of (2). Combining (7a) and (7b), we find

$$\int_{-b}^{x} \psi(y,\nu) e^{-\frac{x-y}{\mu}} dy = \frac{\mu}{\nu-\mu} [\nu\psi(x,\nu) - \mu\psi(x,\mu)] + \frac{\mu}{\nu-\mu} e^{-\frac{b+x}{\mu}} [\mu\psi(-b,\mu) - \nu\psi(-b,\nu)] . \quad (8)$$

Similarly,

$$\int_{-b}^{x} \psi(y,-\nu) e^{-\frac{x-y}{\mu}} dy = \frac{\mu}{\nu+\mu} [\nu\psi(x,-\nu) + \mu\psi(x,\mu)] - \frac{\mu}{\nu+\mu} e^{-\frac{b+x}{\mu}} [\nu\psi(b,\nu) + \mu\psi(-b,\mu)]. \quad (9)$$

Both ν and μ lie in the interval [0,1]. Observe that the right side of (8) is not singular, since both numerator and denominator vanish at $\mu = \nu$. The integration of each term separately is a Cauchy principal value and will be denoted by putting P in front of the integral.

Substituting (8) and (9) into (6) and collecting similar terms with the aid of the symmetry condition of (4), we finally arrive at the singular integral equation

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[1] G. J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," Argonne National Laboratory Report, ANL-6787 (November, 1963).

[2] A. LEONARD and T. W. MULLIKIN, "Solutions to the Criticality Problems for Spheres and Slabs," *The Rand Corporation Memorandum*, RM-3256-PR, (July, 1962).

$$\lambda(\mu)\psi(x,\mu) - P\int_{-1}^{1} \frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)}{\nu-\mu}\,d\nu =$$

$$e^{-\frac{b+x}{\mu}}\left[\lambda(\mu)\psi(-b,\mu) - P\int_{-1}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(-b,\nu)}{\nu-\mu}d\nu\right]; \mu>0, \quad (10)$$

where

$$\lambda(\mu) = \frac{1}{2}\left[\Lambda^{+}(\mu) + \Lambda^{-}(\mu)\right] = 1 - c\mu \tanh^{-1}\mu, \quad (10a)$$

and $\Lambda^{\pm}(\mu)$ are the boundary values of the function

$$\Lambda(z) = 1 - \frac{c}{2}z\int_{-1}^{1}\frac{d\nu}{z-\nu} = 1 - cz\tanh^{-1}\frac{1}{z} \quad (11)$$

as it approaches the cut $[-1,1]$ from above and below the real axis, respectively.

If the boundary condition requires that the incoming distribution is zero:

$$\psi(-b,\mu>0) = \psi(b,\mu<0) = 0\,, \quad (12)$$

Equation (10) reduced to

$$\lambda(\mu)\psi(x,\mu) - P\int_{-1}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)}{\nu-\mu}d\nu$$

$$= -e^{-x/\mu}f(b,\mu);\ \mu>0, \quad (13)$$

where we have defined

$$f(b,\mu) = e^{-b/\mu}\int_{0}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(b,\nu)}{\nu+\mu}\,d\nu\,. \quad (14)$$

The singular equation for $\psi(x,\mu<0)$ also follows from the foregoing results by symmetry.

Now consider

$$\psi(x,z) = \frac{c}{2}\frac{1}{z}\int_{-b}^{x}\rho(y)\,e^{-\frac{x-y}{z}}\,dy;\quad Re(z)>0, \quad (15)$$

which is recognized as the extension of (2), subject to (12), to the right half of the complex plane. The analysis leading to (10) gives, in this case,

$$\Lambda(z)\psi(x,z) = \int_{-1}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)dz}{\nu-z} - e^{-x/z}f(b,z). \quad (16)$$

Since the left side of (16) vanishes at the roots $\pm\nu_0$ of $\Lambda(z)=0$, the right side must also vanish at these two points. Hence we have the two conditions

$$\int_{-1}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(x,\nu)}{\nu\mp\nu_0}d\nu = e^{\mp x/\nu_0}f(b,\pm\nu_0). \quad (17)$$

Setting $x=b$ in (13) and (17), we get

$$\lambda(\mu)\psi(b,\mu) - P\int_{0}^{1}\frac{\left(\frac{c}{2}\right)\nu\psi(b,\nu)}{\nu-\mu}d\nu = -e^{b/\mu}f(b,\mu), \quad (18)$$

and

$$f(b,\nu_0) = f(b,-\nu_0), \quad (19)$$

respectively. These last two equations for the emerging distribution correspond, apart from notation, to the results derived, in a somewhat different way, by Leonard and Mullikin[2].

The advantage of this transformation arises from the fact that a general method for treating singular integral equations is available[3]. Consequently, (10) is a simpler starting point than the original transport equation (1). Moreover, this transformation provides a different way of treating transport problems in this geometry from the method of singular expansion modes developed by Case[4]. What is more important, the success of this method in plane geometry serves as the prime motivation for investigating the transform properties of the integral equations in spherical and cylindrical geometries where the eigenfunction expansions become intractable[1]. The application of this approach to the one-dimensional critical problem, as well as a comparison with the results obtained previously by means of Case's formulation[5], is given in Reference 1.

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Received December 19, 1963

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Effective Surface in Lattices in the Calculation of Resonance Integrals

The exact and approximate method for the calculation of the Dancoff Factor (C) has been examined by many people[1-3]. However, little attention has yet been paid to the relation between the effective surface (S_{eff}) and the Dancoff Factor.

Recently, Levine[4,5] obtained a new relation empirically from a Monte Carlo study and has discussed it in terms of the Bell approximation for the collision probability. His expression is

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