TABLE I

Few-Mode Approximation [Eq. (9)] vs Point Kinetics and Exact Solution

Reactivity Perturbation[a] B^2/B_1^2	Maximum Energy Density at Slab Center (Arbitrary Units)[b]					
	Exact	Point Kinetics (One-Mode Approx.)	Two-Mode Approx.	Three-Mode Approx.	Four-Mode Approx.	Five-Mode Approx.
1.2	0.235	0.236	0.234	0.234	0.234	0.234
2	1.162	1.178	1.144	1.146	1.146	1.146
3	2.297	2.356	2.199	2.205	2.204	2.204
4	3.408	3.534	3.111	3.125	3.123	3.123
7	6.640	7.069	2.832	2.897	2.887	2.889
8	7.693	8.247	-3.295	-3.200	-3.214	-3.211
10	9.774	10.603	29.681	29.861	29.839	29.843

[a]The normalized slab thickness is π, so that the critical buckling is $B_1^2 = 1$. Note that, Eq. (9a), for a step increase in the material buckling $B^2/B_1^2 > 9$, not only the fundamental mode but also the first harmonic is excited with a positive reactivity. Therefore, the last excursion in the table is extremely violent.

[b]For simplicity in Eqs. (9), we took $(\alpha/2) = 1$, where α has units of energy^{-1} times length^{-1}.

The reactivities and temperature coefficients are given by

$$\rho_\nu \equiv M^2(B^2 - B_\nu^2) \quad , \quad b_\nu^{11} \equiv M^2 \alpha B_\nu^{11} \quad , \tag{9a}$$

where M^2 is the migration area and B is the slab half-thickness. The first equation of Eq. (9) is the classic Fuchs-Nordheim result, and the E_ν's are the higher mode contributions to the final energy distribution in the sense of Eq. (5) and the approximation (3).

In Table I, the results obtained from point kinetics (fundamental mode only) and a few mode calculations are shown together with the exact values.[3] It is seen that the approximation (3) gives a fairly good result for the relatively mild excursions where $B^2/B_1^2 < 4$ (ratio of perturbed buckling to critical buckling of the homogeneous reactor). In the limit of $B^2/B_1^2 \to 1$, the approximation is entirely satisfactory and so is point kinetics also. However, for the more severe excursions, the higher mode approximations lead to poorer results than those obtained with the fundamental mode only. The reason is as follows. If for any given time, the flux is written symbolically as

$$\phi(x,t) = A\phi_1(x) + \Delta\phi_1(x,t) \quad , \tag{10}$$

where $\phi_1(x)$ is the fundamental mode shape and $\Delta\phi_1$ the deviation from it, one has

$$[\phi(x,t)]^2 = A^2\phi_1^2 + 2A\phi_1\Delta\phi_1 + \Delta\phi_1^2 \quad . \tag{11}$$

Therefore, if at all times the condition

$$\Delta\phi_1(x,t) << A\,\phi_1(x) \tag{12}$$

is satisfied everywhere, then the approximation of the nonlinear terms in Eq. (2) by Eq. (3) will be valid. Table I shows clearly that, for the more violent excursions, significant changes in the flux shape take place and, therefore, Eq. (12) is not satisfied. In conclusion, the approximation of the nonlinear terms in Eq. (2) by Eq. (3) is equivalent to a perturbation theory treatment and is, therefore, valid only for relatively mild excursions. For the more violent excursions and for a given number of terms in the modal expansion, the nonlinear term in Eq. (2) must be treated rigorously as was shown in Ref. 3.

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Concerning Comments on Two Papers on the Nonlinear Diffusion Equation

The iteration scheme claimed by Canosa[1] [his Eq. (3)] to have been used by Garabedian and Lynch[2] was in fact *not* the scheme they used. Their iteration scheme is clearly displayed in their Eq. (50) and specific instances of it are displayed in their next seven equations. Canosa[1] discusses only the *first step* [Eq.[1] (4), Eq.[2] (54)] of the iteration scheme used by Garabedian and Lynch. His conclusions, therefore, must be limited to this first step. By an analysis of only the first step, he has *not* (as claimed in paragraph one[1]) shown that Garabedian and Lynch's treatment of the nonlinear term is equivalent to a perturbation theory treatment because he does not discuss the *complete iteration scheme.*

Because Canosa treats only the first step in the iteration, I interpret his main point to be: The larger B^2/B_1^2 is, the larger is the difference between the solution of the differential equation and the estimate of it as given by the *first iterate.* This is, of course, obvious.

On the other hand, the claim in Canosa's last sentence in paragraph one,[1] when applied to the *complete iteration scheme* is *not* at all obvious and it should be substantiated with proof (if it is in fact true). If Canosa cannot supply proof, he should at least state plausible reasons for his conjecture and also clearly label it as a conjecture. (Conversely, Garabedian and Lynch[2] do not supply proof that their scheme converges.)

An analysis of the iteration scheme used by Garabedian and Lynch which demonstrates convergence (or divergence) would be worthy of publication. We now supply a first step toward this and show that the series representation of the *first iterate* converges. We use the notation of Garabedian and Lynch.[2]

Consider

$$\phi^{(1)}(x,t) = \Sigma_\nu T_\nu^{(1)}(t)\cos(B_\nu x) \quad , \quad B_\nu = (2\nu - 1)\,\pi/2L \quad , \tag{1}$$

where

[1]J. CANOSA, "Comments on Two Papers on the Nonlinear Diffusion Equation," *Nucl. Sci. Eng.*, **32**, 156 (1968).

[2]H. L. GARABEDIAN and R. E. LYNCH, *Nucl. Sci. Eng.*, **21**, 550 (1965).

$$\dot{T}_\nu^{(1)}(t) + b_\nu T_\nu^{(1)}(t) = -\alpha_\nu\, g(t) \quad , \tag{2a}$$

$$g(t) = T_1^{(1)}(t) \int_0^t T_1^{(1)}(s)\, ds \quad , \tag{2b}$$

$$b_1 < 0 < b_2 < b_3 < b_4 \ldots , \quad -\alpha_\nu = (-1)^2\, 8/\pi(2\nu - 1) \times (2\nu + 1)\,(2\nu - 3) \quad , \tag{2c}$$

$$T_1^{(1)}(0) = A, \quad T_\nu^{(1)}(0) = 0, \quad \nu = 2, 3, \ldots . \tag{2d}$$

The solution of Eq. (2a) subject to the initial conditions, Eq. (2d), is

$$T_\nu^{(1)}(t) = \begin{cases} A \exp(-b_1 t) - \alpha_1 \int_0^t \exp[-b_1(t-\tau)]\, g(\tau)\, d\tau \quad , \\ -\alpha_\nu \int_0^t \exp[-b_\nu(t-\tau)]\, g(\tau) d\tau \quad , \quad \nu = 2, 3, \ldots . \end{cases} \tag{3}$$

In particular (Garabedian and Lynch[2]),

$$T_1^{(1)}(t) = y_m \operatorname{sech}^2 [(2\alpha_1 y_m)^{1/2}\,(t - t_0)/2] \quad , \tag{4a}$$

$$y_m = (b_1^2 + 2\alpha A)/2\alpha_1 \quad . \tag{4b}$$

Since $b_1 < b_\nu$,

$$\begin{aligned} |T_\nu^{(1)}(t)| &= \left|-\alpha_\nu \int_0^t \exp[-b_\nu(t-\tau)]\, g(\tau) d\tau\right| < \left| -\alpha_\nu \int_0^t \exp[-b_1(t-\tau)]\, g(\tau) d\tau \right| \\ &= \left| \frac{-\alpha_\nu}{-\alpha_1} [\dot{T}_1^{(1)}(t) + b_1\, T_1^{(1)}(t)] \right| \quad . \end{aligned} \tag{5}$$

From Eq. (4), it follows that both $T_1^{(1)}$ and $\dot{T}_1^{(1)}$ are uniformly bounded in $0 \leq t < \infty$. Hence, there is a positive constant K such that

$$|T_\nu^{(1)}(t)| \leq |\alpha_\nu| K \quad \nu = 1, 2, \ldots , \quad 0 \leq t < \infty \quad . \tag{16}$$

Thus

$$|\phi^{(1)}(x,t)| = |\Sigma_\nu T_\nu^{(1)}(t) \cos(B_\nu x)| \leq K\, \Sigma_\nu |\alpha_\nu| \quad , \quad 0 \leq t < \infty \quad , \quad |x| \leq L \quad . \tag{7}$$

From Eq. (2c), we see that $\alpha_\nu = 0(\nu^{-3})$; hence, the series converges at each point in $0 \leq t < \infty$, $|x| \leq L$ and we have the following

THEOREM: The series representation of the first iterate of Garabedian and Lynch [i.e., the right-hand side of Eq. (1)] converges at each point in $0 \leq t < \infty$, $|x| \leq L$.

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Comments on Canosa's Letter Regarding the Nonlinear Diffusion Equation

The Letter by Canosa on the quasilinear modal expansion procedure fails to take cognizance of the full set of equations necessary for describing violent excursions in the event that spatial distortions of the flux are important. This set of equations is given in Sec. II of Ref. 1; the numerical effect of these corrections in a particular case is discussed in Sec. IV of the same paper.

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[1] R. SCALETTAR, "Space and Energy-Dependent Corrections to the Fuchs-Nordheim Model," GA-4080, General Atomic (October 7, 1963); abbreviated treatment is given in *Proc. Reactor Kinetics and Control*, University of Arizona, 1963, p. 253, TID-7662, USAEC, Technical Information Division (1964).