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Letters to the Editor

Comments on Two Papers on the Nonlinear Diffusion Equation

In two recent papers on the solution of the one-group nonlinear space-dependent diffusion equation for the neutron flux,

$$\frac{\partial^2\phi(x,t)}{\partial x^2} + B^2\phi - \alpha\left[\gamma \int_0^t \phi(x,\tau)\,d\tau\right]\phi = \frac{1}{Dv}\frac{\partial\phi}{\partial t} \quad , \qquad (1)$$

Scalettar[1] and Garabedian and Lynch[2] used a modal expansion with essentially the same approximation scheme to deal with the nonlinearity. It is shown that this approximation is equivalent to a perturbation theory treatment and, therefore, that it is valid only for mild excursions. For large excursions, the modal expansion series used diverges from the exact solution.

In Eq. (1), B^2 is the homogeneous material buckling after the step increase in the infinite multiplication constant producing the excursion, the expression in the square brackets is the energy release, and α is a constant feedback coefficient. When the flux is expanded in the form

$$\phi(x,t) = \sum_{\nu=1}^{\infty} T_\nu(t)\,\phi_\nu(x) \quad ,$$

where ϕ_ν are the symmetric eigenfunctions of the Helmholtz equation for the slab, i.e.,

$$\phi_\nu(x) = \cos B_\nu x \quad , \quad B_\nu = (2\nu - 1)\frac{\pi}{2R} \quad , \quad \nu = 1, 2, 3 \ldots \quad ,$$

Eq. (1) becomes:

$$\sum_\nu \left[-\frac{1}{Dv}\phi_\nu(x)\dot{T}_\nu(t) + (B^2 - B_\nu^2)\phi_\nu T_\nu\right] = \gamma\alpha \sum_\nu \phi_\nu T_\nu \sum_k \phi_k \int_0^t T_k(\tau)\,d\tau \quad . \qquad (2)$$

Scalettar and Garabedian and Lynch now approximate the nonlinear term in Eq. (2) as follows:

$$\sum_{\nu=1}^{\infty}\left[-\frac{1}{Dv}\phi_\nu\dot{T}_\nu + (B^2 - B_\nu^2)\phi_\nu T_\nu\right] \approx \gamma\alpha\,\phi_1^2(x)\,T_1(t)\int_0^t T_1(\tau)\,d\tau \quad . \qquad (3)$$

Multiplying Eq. (3) by ϕ_ν and integrating over the slab, the following system of ordinary differential equations is obtained for the functions $T_\nu(t)$:

$$\begin{aligned} \nu = 1 \qquad & -\frac{1}{Dv}\dot{T}_1 + (B^2 - B_1^2)T_1 = \gamma\alpha B_1^{11} T_1 \int_0^t T_1\,d\tau \\ \nu > 1 \qquad & -\frac{1}{Dv}\dot{T}_\nu + (B^2 - B_\nu^2)T_\nu = \gamma\alpha B_\nu^{11} T_1 \int_0^t T_1\,d\tau \quad , \end{aligned} \qquad (4)$$

where

$$B_\nu^{11} \equiv \frac{1}{R}\int_{-R}^{R}\phi_1^2\phi_\nu\,dx = \frac{(-1)^{\nu+1}}{\pi}\left(\frac{1}{3-2\nu} + \frac{2}{2\nu-1} - \frac{1}{1+2\nu}\right) \quad .$$

To check the approximation (3), the spatial distribution of the energy release in the excursion is now calculated from Eq. (4) and compared with the exact numerical solution of the problem.[3] Following Ergen,[4] we integrate Eq. (4) over all time, noting that, by definition,

$$E(x,t) = \gamma\int_0^t \phi(x,\tau)\,d\tau = \gamma\sum_\nu \phi_\nu(x)\int_0^t T_\nu(\tau)\,d\tau \equiv \sum_\nu E_\nu(t)\,\phi_\nu(x) \quad . \qquad (5)$$

Also, at the end of the excursion the flux is zero everywhere,[3] i.e.,

$$T_\nu(\infty) = 0 \quad , \qquad (6)$$

while at the start the flux is given by the fundamental mode only[2]

$$T_1(0) = A \qquad T_\nu(0) = 0 \quad . \qquad (7)$$

The integration of the left-hand side of system (4) is obvious using Eqs. (5), (6), and (7). The nonlinear term gives

$$\int_0^\infty T_1\left(\int_0^t T_1\,d\tau\right)dt = \frac{1}{\gamma^2}\int_0^\infty E_1\,dE_1 = \frac{1}{2\gamma^2}E_1^2(\infty) \quad , \qquad (8)$$

having used the definition (5). For simplicity, we neglect the inhomogeneous term arising from the integration of the left-hand side of the first equation of Eq. (4); this is equivalent to assuming a negligibly small initial flux level [i.e., in Eq. (7), $A \approx 0$]. Finally, we obtain in closed form the modal coefficients for the energy distribution at the end of the excursion:

$$E_1(\infty) = \frac{2\rho_1}{b_1^{11}} \quad , \qquad E_\nu(\infty) = \frac{b_\nu^{11}}{2\rho_\nu}E_1^2 \quad . \qquad (9)$$

[1] R. SCALETTAR, "Space and Energy-Dependent Corrections to the Fuchs-Nordheim Model," in *Proc. Reactor Kinetics and Control*, University of Arizona, 1963, p. 253, TID-7662, USAEC, Technical Information Division (1964).

[2] H. L. GARABEDIAN and R. E. LYNCH, *Nucl. Sci. Eng.*, **21**, 550 (1965).

[3] J. CANOSA, *Trans. Am. Nucl. Soc.*, **10**, 247 (1967).

[4] W. K. ERGEN, *Trans. Am. Nucl. Soc.*, **8**, 221 (1965).

TABLE I

Few-Mode Approximation [Eq. (9)] vs Point Kinetics and Exact Solution

Reactivity Perturbation[a] B^2/B_1^2	Maximum Energy Density at Slab Center (Arbitrary Units)[b]					
	Exact	Point Kinetics (One-Mode Approx.)	Two-Mode Approx.	Three-Mode Approx.	Four-Mode Approx.	Five-Mode Approx.
1.2	0.235	0.236	0.234	0.234	0.234	0.234
2	1.162	1.178	1.144	1.146	1.146	1.146
3	2.297	2.356	2.199	2.205	2.204	2.204
4	3.408	3.534	3.111	3.125	3.123	3.123
7	6.640	7.069	2.832	2.897	2.887	2.889
8	7.693	8.247	-3.295	-3.200	-3.214	-3.211
10	9.774	10.603	29.681	29.861	29.839	29.843

[a]The normalized slab thickness is π, so that the critical buckling is $B_1^2 = 1$. Note that, Eq. (9a), for a step increase in the material buckling $B^2/B_1^2 > 9$, not only the fundamental mode but also the first harmonic is excited with a positive reactivity. Therefore, the last excursion in the table is extremely violent.

[b]For simplicity in Eqs. (9), we took $(\alpha/2) = 1$, where α has units of energy^{-1} times length^{-1}.

The reactivities and temperature coefficients are given by

$$\rho_\nu \equiv M^2(B^2 - B_\nu^2) \quad , \quad b_\nu^{11} \equiv M^2 \alpha B_\nu^{11} \quad , \tag{9a}$$

where M^2 is the migration area and B is the slab half-thickness. The first equation of Eq. (9) is the classic Fuchs-Nordheim result, and the E_ν's are the higher mode contributions to the final energy distribution in the sense of Eq. (5) and the approximation (3).

In Table I, the results obtained from point kinetics (fundamental mode only) and a few mode calculations are shown together with the exact values.[3] It is seen that the approximation (3) gives a fairly good result for the relatively mild excursions where $B^2/B_1^2 < 4$ (ratio of perturbed buckling to critical buckling of the homogeneous reactor). In the limit of $B^2/B_1^2 \to 1$, the approximation is entirely satisfactory and so is point kinetics also. However, for the more severe excursions, the higher mode approximations lead to poorer results than those obtained with the fundamental mode only. The reason is as follows. If for any given time, the flux is written symbolically as

$$\phi(x,t) = A\phi_1(x) + \Delta\phi_1(x,t) \quad , \tag{10}$$

where $\phi_1(x)$ is the fundamental mode shape and $\Delta\phi_1$ the deviation from it, one has

$$[\phi(x,t)]^2 = A^2\phi_1^2 + 2A\phi_1\Delta\phi_1 + \Delta\phi_1^2 \quad . \tag{11}$$

Therefore, if at all times the condition

$$\Delta\phi_1(x,t) << A\,\phi_1(x) \tag{12}$$

is satisfied everywhere, then the approximation of the nonlinear terms in Eq. (2) by Eq. (3) will be valid. Table I shows clearly that, for the more violent excursions, significant changes in the flux shape take place and, therefore, Eq. (12) is not satisfied. In conclusion, the approximation of the nonlinear terms in Eq. (2) by Eq. (3) is equivalent to a perturbation theory treatment and is, therefore, valid only for relatively mild excursions. For the more violent excursions and for a given number of terms in the modal expansion, the nonlinear term in Eq. (2) must be treated rigorously as was shown in Ref. 3.

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Concerning Comments on Two Papers on the Nonlinear Diffusion Equation

The iteration scheme claimed by Canosa[1] [his Eq. (3)] to have been used by Garabedian and Lynch[2] was in fact *not* the scheme they used. Their iteration scheme is clearly displayed in their Eq. (50) and specific instances of it are displayed in their next seven equations. Canosa[1] discusses only the *first step* [Eq.[1] (4), Eq.[2] (54)] of the iteration scheme used by Garabedian and Lynch. His conclusions, therefore, must be limited to this first step. By an analysis of only the first step, he has *not* (as claimed in paragraph one[1]) shown that Garabedian and Lynch's treatment of the nonlinear term is equivalent to a perturbation theory treatment because he does not discuss the *complete iteration scheme.*

Because Canosa treats only the first step in the iteration, I interpret his main point to be: The larger B^2/B_1^2 is, the larger is the difference between the solution of the differential equation and the estimate of it as given by the *first iterate.* This is, of course, obvious.

On the other hand, the claim in Canosa's last sentence in paragraph one,[1] when applied to the *complete iteration scheme* is *not* at all obvious and it should be substantiated with proof (if it is in fact true). If Canosa cannot supply proof, he should at least state plausible reasons for his conjecture and also clearly label it as a conjecture. (Conversely, Garabedian and Lynch[2] do not supply proof that their scheme converges.)

An analysis of the iteration scheme used by Garabedian and Lynch which demonstrates convergence (or divergence) would be worthy of publication. We now supply a first step toward this and show that the series representation of the *first iterate* converges. We use the notation of Garabedian and Lynch.[2]

Consider

$$\phi^{(1)}(x,t) = \Sigma_\nu T_\nu^{(1)}(t)\cos(B_\nu x) \quad , \quad B_\nu = (2\nu - 1)\,\pi/2L \quad , \tag{1}$$

where

[1]J. CANOSA, "Comments on Two Papers on the Nonlinear Diffusion Equation," *Nucl. Sci. Eng.*, **32**, 156 (1968).

[2]H. L. GARABEDIAN and R. E. LYNCH, *Nucl. Sci. Eng.*, **21**, 550 (1965).