Boeuf and Tassan have measured the hyperfine structure in a similar cluster when the pressure tube was filled with organic coolant. In their case, 19 natural-UO_2 rods of 16.2-mm diam are arranged in a hexagonal pattern. The suggested equation is again found to fit the measured distribution within the experimental errors of[2] ± 3%.

In Fig. 2 the experimental flux distribution through the 7-rod cluster is compared with the analytic function. In this case $p_0 = 1$, $p_1 = 1.103$ and a common value of $\xi = 0.098$ cm^{-2} is used. As is seen, the expression can be used with an assumption of a flat flux across the cladding gap (as in Fig. 1) or a continuous flux (as in Fig. 2) or any intermediary condition depending on theoretical or experimental justification.

The suggested expression should be useful not only in reducing the amount of experimentation needed to completely specify the hyperfine structure of a given cluster, but also in extrapolating results from a given cluster to related cases.

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[2]A. BOEUF and S. TASSAN, "A Measurement of the Fine Structure Distribution of the Thermal Flux in Organic Cooled Fuel Elements in a D_2O Moderated Lattice," Internal Ispra Report DPR/400, PN/37 to be published in *Energia Nucleare*.

A Phase Integral Study of Neutron Thermalization in Graphite*

In recent years there has been a considerable amount of research[1-4] on the representation of the scattering operator by a differential operator of second order. Under the above approximation the WKB solution[5] of the diffusion equation was first introduced by Corngold[1]. He applied it to the heavy-gas model and obtained very good agreement with the 'exact' calculations. The present work is devoted to WKB calculations of the thermal-neutron spectra and other related quantities in graphite.

If the scattering operator L, defined by

$$L\Phi \equiv \int_0^\infty d\epsilon' \left[\Sigma_s(\epsilon' \to \epsilon)\Phi(\epsilon') - \Sigma_s(\epsilon \to \epsilon')\Phi(\epsilon)\right],$$

$$\epsilon = \frac{E}{kT},$$

is represented by a differential operator of second order, and if the neutron conservation and the detailed balance conditions are satisifed[1,2], we get

$$L = \frac{d}{d\epsilon} P(\epsilon) \left[\epsilon \frac{d}{d\epsilon} + \epsilon - 1\right].$$

If we substitute this for the scattering operator in the source-free diffusion equation and inquire about solutions of the form

$$\Phi(\epsilon, \underline{r}, t) = \exp[i\underline{B}\cdot\underline{r} + \lambda t]\Phi(\epsilon)$$

we get

$$\frac{d^2\Psi}{d\epsilon^2} + k^2(\epsilon)\Psi = 0, \tag{1}$$

where $\Psi(\epsilon) = M^{-1/2} p^{1/2} \Phi(\epsilon)$

$$-k^2(\epsilon) = \frac{1}{4}\left(\frac{q}{p}\right)^2 + \frac{1}{2}\frac{d}{d\epsilon}\left(\frac{q}{p}\right) + \left[\frac{\lambda + \lambda_a f(\epsilon)}{v_0\sqrt{\epsilon}} + D(\epsilon)B^2\right]\frac{1}{p(\epsilon)},$$

in which $\frac{\lambda_a f(\epsilon)}{v_0\sqrt{\epsilon}} = \Sigma_a(\epsilon)$

$p(\epsilon) = \epsilon P(\epsilon)$

$M(\epsilon)$ is the Maxwellian distribution $\epsilon e^{-\epsilon}$

$$q(\epsilon) = \frac{1}{M(\epsilon)}\frac{d}{d\epsilon}\left[p(\epsilon)M(\epsilon)\right]$$

D is the diffusion coefficient.

and $\Phi(\epsilon)$ is the neutron flux.

If we solve the above eigenvalue equation for given values of B^2 and $\Sigma_a(\epsilon)$ and seek the eigenvalues and the corresponding eigenfunctions, we get quantities of interest for the interpretation of pulsed neutron experiment. Further, if we set λ=0 and solve Eq. (1) with an appropriate source term, we get the steady state spectra with leakage

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[1]N. CORNGOLD, *Proc. of the Brookhaven Conf. on Neutron Thermalization* (BNL-719), Vol. IV, 1075 (1962).
[2]G. W. SCHAEFER and K. ALLSOPP, *Proc. BNL Conf.*, Vol. II, 614 (1962).
[3]H. H. W. PITCHER, "The Generalized Heavy Free Gas Thermalization Operator," AEEW-M350 (1963).
[4]M. CADILHAC *et al.*, *Proc. BNL Conf.*, II, 439 (1962).

[5]See, for example, P. M. MORSE and H. FESHBACH, "Methods of Theoretical Physics," McGraw-Hill, New York (1953); or J. HEADING, "Phase Integral Methods," John Wiley, New York (1962).

represented by the DB^2 term. The WKB solutions of Eq. (1) are given in Ref. 1.

It is to be noticed that if the assumption of representing the scattering operator by a differential operator of second order is valid, then the knowledge of only one function $P(\epsilon)$ will enable us to solve the entire problem. Various methods have been developed by Schaefer and Allsopp[2] for calculating $P(\epsilon)$, which fall in two categories:

1) The first method consists in associating $P(\epsilon)$ with some basic properties of the moderator, such as the first energy-transfer moment. For graphite, at low energies ($\epsilon \lesssim 3$), $P(\epsilon)$ has been calculated[3] from the following relation[1,2]:

$$P(\epsilon) = \epsilon^{-2} e^{\epsilon} \int_0^{\epsilon} \epsilon' e^{-\epsilon'} M_1(\epsilon') d\epsilon',$$

where $M_1(\epsilon')$ is the first energy-transfer moment calculated from the Egelstaff's model[6] Corngold's asymptotic formula has been used for calculating $P(\epsilon)$ at high energies[2]. (The asymptotic behavior of $P(\epsilon)$ for $\epsilon \to 0$ and $\epsilon \to \infty$ for a crystalline moderator has been discussed by Corngold[1].)

2) The second method consists in calculating $P(\epsilon)$ by using Eq. (1) with the knowledge of the steady-state spectra for a particular system. Schaefer and Allsopp[2] have calculated $P(\epsilon)$ from the experimentally measured spectra in a graphite-moderated (Calder Hall) lattice. (It is to be noted that in this method $P(\epsilon)$ depends on the absorption of the system.)

It should be pointed out that when $P(\epsilon)$ is calculated from the knowledge of a steady-state spectra, the good predictions of $\Phi(\epsilon)$ will not be surprising. It is essentially calculating $P(\epsilon)$ from one system and applying to other similar systems. However, when $P(\epsilon)$ is calculated from models based on scattering law and then the spectrum calculations are done (as is done for graphite), the good agreement of the predicted $\Phi(\epsilon)$ with experiments precisely points out the important properties of the moderator (condensed in $P(\epsilon)$) that determine the spectrum.

The calculated values of $P(\epsilon)$ for 300 and 600°K were obtained from Ref. 3 and fitted within 5% with various analytical formulae, and Eq. (1) was solved using the WKB method. The function $\Delta \left(= \frac{\lambda_0}{\xi \Sigma_f}, \Sigma_f \text{ is the free-atom cross section}\right)$ was least-squares fitted to a power series in $\frac{DB^2}{\xi \Sigma_f}$, and using $D = 0.8645$ cm^{-1} and $\xi \Sigma_f = 0.0595$ cm^{-1}, we obtained the following values of the diffusion parameters: $D_0 = 2.26 \times 10^5$ cm^2/sec and $C = 1.0 \times 10^6$ cm^4/sec. Since the diffusion coefficient D is assumed to be independent of energy the value of D_0 should have been $\frac{2}{\sqrt{\pi}} \nu_0 D$ ($= 2.15 \times 10^5$ cm^2/sec). The difference indicates the error in the WKB approximation. For the sake of comparison, we note that the values obtained by using the Parks' model[8] are $D_0 = 2.226 \times 10^5$ cm^2/sec and $C = 2.12 \times 10^6$ cm^4/sec. The value of λ_1 obtained was 7100 sec^{-1} as compared to 6900 sec^{-1} obtained using Nelkin's 'M_2 formula' and Parks' model for graphite.

We have also calculated the thermal-neutron spectra for graphite assembly poisoned by a $\frac{1}{v}$ absorber, and compared our calculations with the measurements of Beyster[7]. Typical comparisons are shown in Figs. 1 and 2. The presence of leakage was represented by the DB^2 term. The

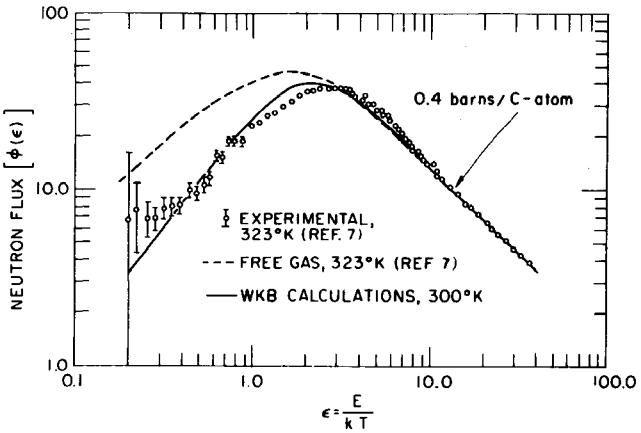

Fig. 1. Thermal-neutron spectra in poisoned graphite at 323 °K.

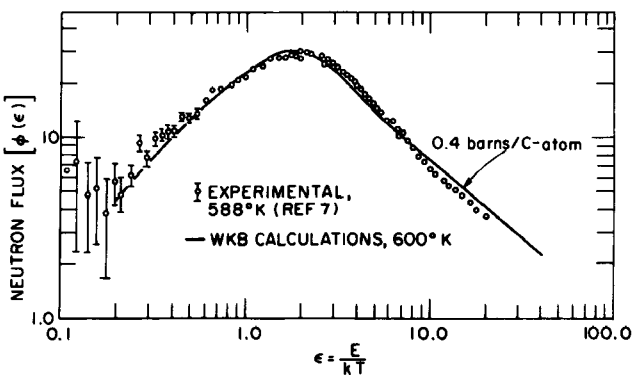

Fig. 2. Thermal-neutron spectra in poisoned graphite at 588 °K.

[6] P. A. EGELSTAFF and P. SCHOFIELD, *Nucl. Sci. Eng.*, **12**, 260 (1962).

[7] J. R. BEYSTER *et al.*, "Integral Neutron Thermalization," General Atomic Report GA-2544 (1961).

[8] H. C. HONECK, *Proc. BNL Conf.*, **IV**, 1186 (1962).

calculated curves are normalized to have the same value of $\int(\Sigma_a + DB^2)\Phi(\epsilon)d\epsilon$ as in the experimental curve. A check of neutron conservation for the calculated spectra has been performed using the typical conservation condition:

$$\int_0^{\epsilon_e} [\Sigma_a + DB^2]\Phi(\epsilon)d\epsilon = q(\epsilon_e) ,$$

where ϵ_e is the thermal cutoff energy above which upscattering is negligible and $q(\epsilon_e)$ is the neutron slowing-down density past ϵ_e which is given by[2]:

$$q(\epsilon_e) = P(\epsilon_e)\,\epsilon_e\,\Phi(\epsilon_e) .$$

For both 300 and 600°K spectra, the conservation condition was satisfied within 5%.

In summary, the WKB method appears to be a simple and flexible method of calculating quantities of interest in neutron thermalization.

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