Letters to the Editor

Comment on "Asymptotic Equivalence of Neutron Diffusion and Transport in Time-Independent Reactor Systems"

In a recent paper, Borysiewicz and Mika[1] used variational principles to derive a diffusion theory as a certain asymptotic limit of transport theory. These authors acknowledge that their small-parameter scaling differs from previous approaches to this problem, and I would like to discuss some important differences (not mentioned in Ref. 1) between the final results in Ref. 1 and previous efforts. First, however, I wish to point out an erroneous statement in Ref. 1 that does a substantial disservice to several earlier researchers.

The authors claim in their introduction to Ref. 1 that "a feature common to all [previous] works is the lack of rigorous proof of the asymptotic convergence." This statement is incorrect. The first rigorous analysis that I am aware of was published in 1975 by Habetler and Matkowsky[2]; their proof, which was given for a one-group slab geometry problem, is based on simple properties of the transport equation. Papanicolaou[3,4] and Bensoussan et al.[5] followed this with some very detailed research based on probabilistic methods and branching processes. Williams[6-8] continued some of this work and has also made use of maximum principles. From a technical point of view, the approach followed by Borysiewicz and Mika,[1] using variational principles, is new. However, a substantial amount of rigorous work has preceded it; moreover, most of this work is discussed in my review article,[9] which is cited in Ref. 1.

The second issue I wish to discuss concerns the fact that in previous work (Ref. 2 is an excellent example), a type of small-parameter scaling is used similar to that in Ref. 1, but only in the interior of the physical system, away from the boundaries. Near the boundaries a different scaling is introduced to account for boundary layers, which describe the transition from the generally anisotropic boundary condition to the isotropic leading-order term of the interior asymptotic expansion. These boundary layers exist unless the boundary conditions happen to precisely match the form of the interior asymptotic solution. These layers are discussed thoroughly in Ref. 2, and also in Ref. 9. Boundary layers occur in many other types of physical phenomena as well.[10] Because Ref. 1 does not include a boundary layer analysis, it seems likely that the results can generally be valid only when boundary layers do not exist, i.e., when the boundary conditions are special. Only in this case would the scaling used in Ref. 1 be valid throughout the system.

As a very simple illustrative example, the problem

$$\mu \frac{\partial \psi}{\partial x} + \psi = \frac{1}{2} \int_{-1}^{1} \psi \, d\mu' \ , \quad 0 < x$$

$$\psi(0,\mu) = f(\mu) \ , \qquad 0 < \mu \leqslant 1$$

has a solution, away from the boundary (i.e., where the boundary layer—or continuum modes—are insignificant), which is constant:

$$\psi(x,\mu) \sim \int_0^1 h(\mu') f(\mu') \, d\mu' \ , \tag{1a}$$

$$h(\mu) = \frac{\mu/X(-\mu)}{\int_0^1 \mu' X(-\mu') \, d\mu'} \ . \tag{1b}$$

(We have used the standard notation of Ref. 11 here.) The theory of Ref. 2 derives this correct result [Eqs. (1)] for all f; whereas the theory of Ref. 1 predicts the (generally incorrect) Marshak[11] result

$$\psi(x,\mu) \sim \int_0^1 2\mu' f(\mu') \, d\mu' \ . \tag{2}$$

[Note that Eqs. (1) and (2) do agree for the special isotropic case $f(\mu) = 1$; here the boundary layers are $O(\epsilon)$ in magnitude.]

To summarize, in the above paragraph, the theory of Ref. 1 gives the correct result that $\psi \sim$ constant, but for problems having an $O(1)$ boundary layer (i.e., for problems with nonisotropic boundary conditions), the Ref. 1 boundary conditions generally have an $O(1)$ error that propagates throughout the entire system producing a global $O(1)$ error. For the energy-dependent problems discussed in Ref. 1, if

[1] M. BORYSIEWICZ and J. MIKA, *Nucl. Sci. Eng.*, **81**, 110 (1982).

[2] G. J. HABETLER and B. J. MATKOWSKY, *J. Math. Phys.*, **16**, 846 (1975).

[3] G. C. PAPANICOLAOU, *Bull. Amer. Math. Soc.*, **81**, 330 (1975).

[4] G. C. PAPANICOLAOU, "Boundary Behavior of Branching Transport Processes," *Stochastic Analysis*, A. FRIEDMAN and M. PINSKY, Eds., Academic Press, New York (1978).

[5] A. BENSOUSSAN, J. L. LIONS, and G. C. PAPANICOLAOU, "Boundary Layers and Homogenization of Transport Processes," *RIMS*, Kyoto University, **15**, 53 (1979).

[6] MICHAEL WILLIAMS, "Homogenization of Linear Transport Problems," Thesis Dissertation, New York University (1976).

[7] MICHAEL WILLIAMS, *Ann. Nucl. Energy*, **7**, 257 (1979).

[8] MICHAEL WILLIAMS, *Prog. Nucl. Energy*, **8**, 95 (1981).

[9] E. W. LARSEN, *Ann. Nucl. Energy*, **7**, 249 (1979).

[10] J. KEVORKIAN and J. D. COLE, *Perturbation Methods in Applied Mathematics*, Springer-Verlag, New York (1980).

[11] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley Publishing Company, Inc., Reading (1967).

the boundary conditions have the form

$$\psi(\boldsymbol{x},\boldsymbol{w}) = A(\boldsymbol{x})m(w) \ , \quad \boldsymbol{x} \in \Gamma \ , \quad \boldsymbol{n}\cdot\boldsymbol{w} < 0 \ , \tag{3}$$

then an $O(1)$ boundary layer does not exist and the analysis of Ref. 1 is probably valid; conversely, if the boundary conditions cannot be written in the form of Eq. (3), then an $O(1)$ boundary layer does exist, and by analogy to the results in the above paragraph, the analysis of Ref. 1 is almost certainly not valid.

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Response to "Comment on 'Asymptotic Equivalence of Neutron Diffusion and Transport in Time-Independent Reactor Systems'"

The statement in our paper[1] concerned with the lack of rigor in previous publications on asymptotic problems in transport theory is, in fact, too strong for two reasons. First, there are apparently papers that treat the asymptotic limit of the solution to the transport equation in a rigorous way (see Refs. 2 through 8 from Ref. 2). Second, no matter how rigorous is the derivation, the asymptotic limit depends on the scaling of the original equation, which, almost inevitably, is of heuristic character.

The last statement might perhaps be an explanation of the differences in the results (and opinions) between us and Larsen.[2] Since our analysis seems to be rigorous (in the asymptotic sense), the result shows that any corrections to the boundary conditions for the diffusion equation obtained by us should be of order $O(\epsilon)$ with the particular scaling proposed in our paper. With a different scaling one can get another boundary condition. The two can differ from each other by terms of order $O(1)$, but that does not mean that either of the asymptotic analyses is mathematically erroneous, although one or the other might be superior from the physical point of view, depending on the particular problem to be studied.

Summarizing, we claim that with our scaling of the transport equation, the boundary condition for the diffusion equation derived in our paper takes properly into account the boundary layer up to terms of order $O(\epsilon)$. However we did not consider[1] whether there are cases of practical importance for which our assumptions are physically justified, although the answer seems to be positive, even in conditions far from criticality. In the example considered by Larsen,[2] the absorption term is taken to be identically equal to zero and the maximum time for a neutron to travel across the medium is infinite. This combination is excluded from our analysis, so it is impossible to compare the two approaches in that physical situation. Nevertheless, we think that the asymptotic analysis of the transport equation is very important, both from theoretical and physical points of view, and we hope that our paper has contributed in one way or another to this subject.

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[1]M. BORYSIEWICZ, J. MIKA, and G. SPIGA, *Nucl. Sci. Eng.*, **81**, 110 (1982).

[2]E. W. LARSEN, *Nucl. Sci. Eng.*, **83**, 522 (1983).

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