For an average slip ratio <3.109, $\rho_l - 2\bar{\rho}$ is positive, so increasing slip increases the circulation ratio. Since the conditions assumed here are similar to those from Ref. 1 cited above, the behavior reported there is physically reasonable.

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REFERENCES

1. G. W. HOPKINS, A. Y. LEE, and O. J. MENDLER, "Thermal and Hydraulic Code Verification: ATHOS2 and Model Boiler No. 2 Data," Vol. 1, EPRI NP-2887, Electric Power Research Institute (Feb. 1983).

2. G. S. LELLOUCHE and B. A. ZOLOTAR, "A Mechanistic Model for Predicting Two-Phase Void Fraction for Water in Vertical Tubes, Channels and Rod Bundles," EPRI NP-2246-SR, Electric Power Research Institute (1982).

3. L. W. KEETON et al., "The URSULA2 Computer Program, Vol. 2: Applications (Sensitivity Studies and Demonstration Calculations)," EPRI NP-1315, Electric Power Research Institute (Jan. 1983). Used by permission.

Richardson Extrapolation

The only aim of the present Letter is to draw the readers' attention to an old method[1] for increasing the accuracy of numerical solutions of linear problems. Customarily, new finite element (FE) or coarse-mesh (CM) algorithms are checked on well-defined benchmark problems,[2] and the reference solution of a given benchmark problem is usually obtained by a well-established program using a large number of meshes.

We are concerned here with the Richardson extrapolation,[3] which allows one to obtain higher accuracy without refining further the mesh. The method can be used independently of the geometry.

It is well known that the accuracy of a numerical solution is often proportional to some power of the mesh size h. The accuracy of the finite difference (FD) method is $O(h)$, that of the linear FE method is $O(h^2)$. The basic idea in Richardson extrapolation is to separate a part of the error, proportional to some power of h, and to eliminate it. Let us consider the problem to be solved as

$$Lu = f \quad \text{in} \quad \Omega \ , \tag{1a}$$

$$Bu = g \quad \text{on} \quad \partial\Omega \ , \tag{1b}$$

where $\partial\Omega$ is the boundary of the region Ω. In numerical methods Eq. (1) is substituted by the discretized formulas

$$L_h u_h = f_h \quad \text{in} \quad \Omega_h \tag{2a}$$

$$B_h u_h = g_h \quad \text{on} \quad \partial\Omega_h \ . \tag{2b}$$

In the discretized formulas the dependence on the mesh size h is indicated explicitly. The discretized form of the most frequently used operators is available in handbooks (see Ref. 4). Let us assume Eq. (2) to have a unique solution that is sufficiently smooth.[3] The solution of Eq. (2) then has the following form:

TABLE I

Richardson Extrapolation of BUG-180 Results, Using 3 and 12 Points per Hexagon, Problem GA9A1

Number	B3[a]	B12[a]	RE[b]	B48[a]
1	0.3798	0.3755	0.3741	0.3745
2	0.9988	1.0497	1.0667	1.0655
3	0.7781	0.8207	0.8349	0.8338
4	0.1891	1.2247	1.2366	1.2349
5	1.2358	1.2777	1.2917	1.2902
6	1.2183	1.2662	1.2821	1.2806
7	1.2450	1.2906	1.3058	1.3043
8	1.1999	1.2404	1.2539	1.2522
9	1.2147	1.2470	1.2578	1.2562
10	0.7442	0.7638	0.7703	0.7695
11	1.1766	1.1692	1.1667	1.1665
12	1.1328	1.1431	1.1465	1.1455
13	1.1938	1.2149	1.2219	1.2207
14	1.1565	1.1761	1.1826	1.1812
15	1.1522	1.1551	1.1561	1.1552
16	1.1802	1.1699	1.1665	1.1663
17	0.3494	0.3374	0.3334	0.3339
18	0.8565	0.8673	0.8709	0.8703
19	0.9093	0.9284	0.9348	0.9338
20	0.9547	0.9879	0.9990	0.9977
21	0.9665	1.0048	1.0176	1.0162
22	0.9419	0.9702	0.9796	0.9785
23	0.8859	0.9007	0.9056	0.9048
24	0.7534	0.7280	0.7195	0.7206
25	1.0347	0.9833	0.9662	0.9688
26	1.1301	1.1012	1.0916	1.0928
27	1.1038	1.0702	1.0590	1.0605
28	0.9614	0.9063	0.8879	0.8907
29	0.6444	0.6412	0.6401	0.6404
30	0.9144	0.8673	0.8576	0.8541
31	0.9867	0.9557	0.9454	0.9467
32	1.0122	1.0052	1.0029	1.0028
33	1.0913	1.0852	1.0832	1.0830
34	1.1010	1.0745	1.0657	1.0668
35	0.9778	0.9306	0.9149	0.9174
36	0.6438	0.6409	0.6399	0.6402
37	0.9128	0.8654	0.8496	0.8520
38	1.0073	0.9696	0.9570	0.9587
39	1.1131	1.0972	1.0919	1.0921
40	1.0477	1.0454	1.0446	1.0441
41	1.0215	1.0056	1.0003	1.0008
42	0.9655	0.9256	0.9123	0.9141
Time[c]	1.92	13.27	15.19	160.9

[a]Bn: BUG-180 result using n points per hexagon.
[b]RE: Richardson extrapolation.
[c]UNIVAC 1108 CPU min.

$$u_h = u + \sum_{j=1}^{m} v_j h^j + \eta_h \ , \qquad (3)$$

where

u = exact solution of Eq. (1)

h = mesh size

v_j = functions independent of h

η_h = so-called remainder

m = integer connected with the smoothness[5] of functions f and g.

Let us form a linear expression from the approximate solutions u_{h_k} associated with mesh size h_k as follows:

$$V = \sum_{k=1}^{m+1} \gamma_k u_{h_k} \qquad (4)$$

and the weighting γ_k is obtained from

$$\sum_{k=1}^{m+1} \gamma_k = 1 \qquad (5)$$

$$\sum_{k=1}^{m+1} \gamma_k \cdot (h_k)^j = 0 \ , \quad j = 1, \ldots, m \ . \qquad (6)$$

For example, if $h_1 = H$ and $h_2 = H/2$ then $\gamma_1 = -1/3$; $\gamma_2 = 4/3$. The following estimation is valid for V:

$$V - u = \sum_{k=1}^{m+1} \gamma_k \cdot \eta_{h_k} \ . \qquad (7)$$

Let us remark that the error of V does not include any part proportional to some power of the mesh size. An estimation of the functions η_{h_k} is available through the smoothness of the coefficients figuring in the original operator L. To take an example, in the one group diffusion equation, let the coefficients belong[a] to $C^r[0,1]$, in which case the estimation

$$|V - u| \leq c \cdot h^r \qquad (8)$$

holds. Usually $r > 2$ so the error of the approximate solution V is smaller than the error of any u_{h_k}.

To illustrate the method let us consider a high-temperature gas-cooled reactor benchmark[2] identified as GA9A1. In Table I the FD solutions by the code BUG-180 using 3 and 12 points per hexagon and the Richardson extrapolated solution are compared with the solution using 48 points per hexagon. The applied weightings are

$$V = -\frac{1}{3} u_h + \frac{4}{3} u_{h/2} \ , \qquad (9)$$

where $h = 20.9$ cm. As we can see, the Richardson extrapolation has improved the accuracy considerably. It is pointed out that from a VENTURE-like[6] FD solution one can obtain a solution in ~1 min [IBM 360/195 central processing unit (CPU)] that is not inferior in accuracy to CM or FE solutions. The author is convinced that opinions[7] regarding the outstanding efficiency of CM and FE methods over FD methods should be revised.

[a] $C^r[0,1]$ denotes the set of functions which are r times continuously differentiable on [0,1].

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REFERENCES

1. L. F. RICHARDSON, *Philos. Trans. Roy. Soc., London, Ser. A.*, **210**, 307 (1910).

2. *Benchmark Problem Book*, ANL-7416, Suppl. 2, Argonne National Laboratory (1977).

3. G. I. MARCHUK and V. V. SHAIDUROV, *Difference Methods and Their Extrapolations*, Springer Verlag, New York (1983).

4. G. A. KORN and T. M. KORN, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Book Company, New York (1970).

5. For further details see Ref. 3, p. 16.

6. D. R. VONDY and T. B. FOWLER, "VENTURE," ORNL-5062, Oak Ridge National Laboratory (1975).

7. R. D. LAWRENCE and J. J. DORNING, *Nucl. Sci. Eng.*, **77**, 502 (1981), express this rather general opinion; as evidence that the present author was of a similar opinion, see M. MAKAI and C. MAEDER, *Nucl. Sci. Eng.*, **84**, 390 (1983).