as the odd powers of $\bar{Z}_i$ have zero expectation according to Eq. (8). Finally, since from a statistical point of view, all the $\bar{Z}_i$'s are equivalent, we have

$$\langle \bar{Z}_i^r \rangle = \langle \bar{Z}_1^r \rangle = \left\langle \left(\frac{1}{k} \sum_{i=1}^{k} y_i \right)^r \right\rangle \tag{9}$$

and

$$Q(k) = \frac{n^2}{p^3} \langle \bar{Z}_1^4 \rangle + \frac{n^2[(p-1)^2+1]}{p^3(p-1)} \langle \bar{Z}_1^2 \rangle^2 \ . \tag{10}$$

It remains to determine the expectations in Eq. (9) for $r = 2$ and $r = 4$. For $r = 2$ the result is commonplace:

$$\langle \bar{Z}_1^2 \rangle = \left\langle \left(\frac{1}{k} \sum_{i=1}^{k} y_i \right)^2 \right\rangle = \left\langle \frac{1}{k^2} \sum_{i=1}^{k} y_i^2 \right\rangle = \frac{1}{k} \langle y^2 \rangle \ . \tag{11}$$

Similarly for the fourth moment,

$$\begin{aligned} \langle \bar{Z}_1^4 \rangle &= \left\langle \left(\frac{1}{k} \sum_{i=1}^{k} y_i \right)^4 \right\rangle = \frac{1}{k^4} \left\langle \left(\sum_{i=1}^{k} y_i^2 + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} y_i y_j \right)^2 \right\rangle \\ &= \frac{1}{k^4} \left\langle \sum_{i=1}^{k} y_i^4 + 2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\neq i}}^{k} y_i^2 y_j^2 \right\rangle \\ &= \frac{1}{k^3} \langle y^4 \rangle + \frac{2(k-1)}{k^3} \langle y^2 \rangle^2 \ . \end{aligned} \tag{12}$$

Inserting Eqs. (11) and (12) into Eq. (10), the quantity to be minimized becomes

$$Q(k) = \frac{1}{n} \left[\langle y^4 \rangle + \left(n - 2 + k + \frac{k^2}{n-k} \right) \langle y^2 \rangle^2 \right] \ , \tag{13}$$

where we have put $p = n/k$. Obviously $Q(k)$ is minimal with $k = 1$; i.e., *the variance of the estimated variance is minimal if every batch consists of a single realization.*

Therefore, when estimating the theoretical variance of a random variable from the realizations $x_1, x_2, \ldots, x_n$, the most efficient estimate follows from Eqs. (4) and (5) as

$$\hat{V} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{S})^2 \ , \tag{14}$$

where $\hat{S}$ is the empirical mean of the realizations as given in Eq. (3).

The minimum value of $Q(k)$ is

$$\begin{aligned} Q(1) &= \frac{1}{n} \left[\langle y^4 \rangle + \left(n - 1 + \frac{1}{n-1} \right) \langle y^2 \rangle^2 \right] \\ &= \frac{1}{n} [\langle (x-\mu)^4 \rangle - \langle (x-\mu)^2 \rangle^2] \\ &\quad + \left[\frac{1}{n(n-1)} + 1 \right] \langle (x-\mu)^2 \rangle^2 \ , \end{aligned}$$

and according to Eq. (6), the variance of the optimal variance estimate is

$$\begin{aligned} \hat{D}^2 &= Q(1) - \langle (x-\mu)^2 \rangle^2 \\ &= \frac{1}{n} [\langle (x-\mu)^4 \rangle - \langle (x-\mu)^2 \rangle^2] + \frac{1}{n(n-1)} \langle (x-\mu)^2 \rangle^2 \ . \end{aligned}$$

Comparing it with Eq. (2), it is apparent that

$$\hat{D}^2 = D^2 + \sigma^4/[n(n-1)] \ ;$$

i.e., *the variance is increased by* $\sigma^4/[n(n-1)]$ *if the mean of the realizations is not known but is also to be estimated.*

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Limitations on the Use of the THOR Critical Assembly for Validation of n + ^{239}Pu Cross Sections

A major revision of the ENDF/B-V evaluation of neutron-induced nuclear data for ^{239}Pu was recently provided by Arthur et al.[1] The revised data were validated by calculating measured quantities for the five fast critical assemblies JEZEBEL, JEZEBEL-PU, FLATTOP-PU, THOR, and ZPR-6/7. The integral parameters calculated are k_{eff} and certain fission ratios. The revised data set improved the agreement between calculated and measured integral parameter values ("the agreement") for all the assemblies except the JEZEBEL-PU assembly. Table I gives the extent of improvement obtained[1] using the revised set for the five assemblies. One can see that the improvement of the agreement is maximum for the THOR assembly and the agreement has worsened in the case of JEZEBEL-PU. The authors[1] have emphasized that while their new inelastic, elastic, and total cross-section results are based on a thorough analysis, the $\bar{\nu}_p(E_n)$ and fission spectrum modifications in their paper are of an interim nature, because in both cases entire data bases were not considered. They have also suggested a new analysis of the resolved and unresolved resonance regions that extends the resolved resonance region to as high an energy as feasible and also an analysis of smooth (n,f) and (n,γ) cross sections that accounts for energy correlations in the data.

The purpose of our letter is to point out that the good improvement in the agreement obtained in the case of the THOR assembly may be fortuitous. The comment is only on the weakness of using the THOR assembly for testing ^{239}Pu cross sections and not on the quality of the evaluation of ^{239}Pu cross sections themselves.

TABLE I

Deviations of k_{eff} from Unity for the Critical Assemblies When ENDF/B-V and Revision 2 Data Sets for ^{239}Pu Are Used

Critical Assembly	ENDF/B-V (%)	Revision 2 (%)
JEZEBEL	0.68	−0.18
JEZEBEL-PU	−0.20	−0.83
FLATTOP-PU	0.93	0.50
THOR	2.28	0.70
ZPR-6/7	−0.44	−0.42

The THOR critical assembly[2] has a large blanket of thorium with an atomic concentration of 0.03005 nucleus/b·cm compared to the atomic concentration of 0.03618 nucleus/b·cm for ^{239}Pu in the relatively small core. This means that cross sections of ^{232}Th should be assumed to be very well known to talk confidently about testing and validation of cross sections of ^{239}Pu. A close look at the cross sections of ^{232}Th is, therefore, needed.

The ENDF/B-V evaluations of total, elastic, capture, fission inelastic, $(n,2n)$, and $(n,3n)$ cross sections for thorium were made by Meadows et al.[3] in the 50-keV to 20-MeV energy region. They found the then existing data base uncertain in a number of areas and felt that future measurement programs should provide quantitative information before their evaluation could be substantially improved. Since then, two more evaluations of ^{232}Th cross sections have been made, one by Poenitz[4] and the other by a Rumanian group.[5] This group also concluded that experimental measurements and hence a reevaluation are to be performed for ^{232}Th.

An analysis of the THOR assembly was made by us with special attention given to thorium cross sections. The sets that we have used are derived from ENDF/B-IV, the recent evaluation of the Rumanian group, i.e., INDL/A-83 (Ref. 5); JENDL-1 (Ref. 6), and INDIAN. The INDIAN set is essentially an evaluation by Indian workers as summarized in Ref. 7. The use of JENDL-2 (Ref. 8) data for ^{232}Th is not expected to change drastically the conclusions of the present study. The results for k_{eff} are given in Table II. The spread in k_{eff} is as much as 3.79%, arising mainly from uncertainties in inelastic scattering data for ^{232}Th. In all our calculations we have used the ENDF/B-IV cross sections for ^{239}Pu in the core. The calculations are with one-dimensional S_{16} transport theory as recommended by the Cross Section Evaluation Working Group.[2] Our contention, however, that thorium cross sections play an important role in k_{eff} calculations has been borne out by the results. Note here that the recent Coupled Fast Reactivity Measurement Facility[9] integral measurements also indicated the need for improving ENDF/B-V capture data for neutron capture by ^{232}Th by 5 to 10% in the 1-keV to 17-MeV energy region. In our current sensitivity studies, however, the influence of this uncertainty in the capture cross section of ^{232}Th on the k_{eff} of the THOR assembly, which emphasizes transport of neutrons in the fission source energy range, is found to be ~0.3%.

One can see from Table IV of Ref. 1 that the improvement in the fission ratio $\sigma_{f(^{238}U)}/\sigma_{f(^{235}U)}$ is nearly 5% when the Revision 2 data were used in place of ENDF/B-V data. Our calculations[10] of spectral indices with various data sets for thorium in the blanket showed a spread of 4% in this fission ratio, reflecting the influence of uncertainties in thorium cross sections.

TABLE II

Calculated k_{eff} of the THOR Assembly When Various Cross-Section Sets Are Used for ^{232}Th in the Blanket*

Set Number	k_{eff}[a]
1. ENDF/B-IV based RRC set	0.9951
2. JENDL-1 based RRC set	0.9943
3. INDL/A-83 based RRC set	1.0244
4. Indian file based RRC set	0.9865

*In all the calculations presented here, the plutonium core is represented by the ENDF/B-IV based RRC set; gallium in a plutonium core is represented by the ENDL-78 based RRC set.

[a]Measured $k_{eff} = 1.000 \pm 0.001$.

In the last few years, many experiments on ^{232}Th have been performed.[11-16] Fission neutron multiplicities with incident neutron energies to 49 MeV have been measured.[11] This experiment has extended the available multiplicity data into the previously unreported 1.1- to 1.3-MeV and 17- to 49-MeV energy regions. An absolute measurement[12] of the capture cross section for the 23-keV region gives a value 10% higher than ENDF/B-V. Sheldon has reviewed the recent experimental and theoretical work performed on inelastic cross sections for ^{232}Th and other actinide nuclei.[13] He has clearly brought out the need for further revision of the inelastic cross sections of ^{232}Th in ENDF/B-V. These recent developments are recalled here to stress that it is a moot point that one can use the THOR assembly for testing ^{239}Pu cross sections.

In summary, we state the following:

1. The k_{eff} of the THOR assembly is equally sensitive to uncertainties in the cross sections of both ^{232}Th and ^{239}Pu. Hence k_{eff} is not currently suited to validating the cross-section data for ^{239}Pu.

2. The spectral indices at the core of the THOR assembly are again affected by ~4% by uncertainties in thorium cross sections. Therefore, an improvement of this magnitude in the calculated spectral indices at the core of the THOR assembly due to improved cross sections of ^{239}Pu should be viewed with caution.

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Response to "Limitations on the Use of the THOR Critical Assembly for Validation of n + ^{239}Pu Cross Sections"

The integral testing of ENDF/B-V ^{239}Pu Revision 2 was undertaken to further validate this evaluated data file. Our philosophy in doing so was calculation of *all* Los Alamos National Laboratory fast critical assemblies sensitive to ^{239}Pu nuclear data so as to provide as wide a spectrum of conditions for data testing as possible. Thus, the validation of the ^{239}Pu Revision 2 evaluated data file did not rest on results obtained from one assembly but from the general improvement achieved for five assemblies. As noted in our paper,[1] "The average eigenvalue for such assemblies is now essentially unity and their scatter has been reduced significantly."

For an isolated consideration of the THOR assembly, we certainly agree that the calculated results are sensitive to evaluated thorium data as well as to ^{239}Pu cross sections. Keshavamurthy and Ganesan have illustrated this point well,[2] although their comparisons could be affected by their use of the ENDF/B-IV data file for ^{239}Pu for which there are known problems. The most conclusive test of nuclear data in the THOR assembly would thus occur through use of improved evaluations for both ^{239}Pu and ^{232}Th.

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