Letter to the Editor

Terminology for Adjoint Equations and Operators

In writing about the use of variational and perturbation methods, one wishes to refer to the process of constructing an adjoint equation and its adjoint operators. It seems at present we do not have established words to describe this process and this Letter is an invitation to the scholarly community to comment and agree, if possible, on acceptable terminology.

I am assuming that none of us would want to perpetrate "adjointise." Another word that has been used is "invert," but the inverse operator has a very particular meaning and I suggest that it should not be misused here. It can also be noted *en passant* that "adjoint" only means "related," and matrix algebra employs the term "adjoint matrix" in a way that is not consistent with our present requirement; indeed that adjoint matrix is a step on the way to finding the true inverse of an algebraic matrix. To find an appropriate terminology, it would seem sensible to examine the operations we are to undertake before naming them.

It will be recollected that the general process involves starting with a forward equation—we use the neutron density field equation as our example—which in a general time and phase-space notation can be written as

$$\frac{\partial N}{\partial t} = M(N)N + S \ ,$$

where $N = N(x,t)$ and appropriate boundary and initial conditions are imposed on N. The operator M, which may include matrix, differential, and integral terms, is shown generally as N-dependent to allow for the nonlinear case.

The variational-perturbation method is concerned with a characteristic of interest, which in general is some weighted integral $\bar{R}$ over the domain of the system of the form

$$\bar{R} = \iint R(N) N \, dx \, dt \ ,$$

where $R(N)$ is also generally N-dependent.

The required operations are bifurcated in the nonlinear case, where we have to distinguish between the importance of a neutron (field quantum) in the mean to the characteristic of interest and the importance of an additional neutron as the change brought about in $\bar{R}$. We can call these the mean and the marginal importance, respectively. The required equations are:

$$\text{mean importance:} \quad \frac{-\partial N^+}{\partial t} \ M^*(N)N^+ + R = 0 \qquad (1)$$

and

$$\text{marginal importance:} \quad \frac{-\partial \psi^+}{\partial t} = \frac{\partial [M(N)N]^*}{\partial N} \times \psi^+ + \frac{\partial [R(N)N)]}{\partial N} \ , \qquad (2)$$

where we use a functional derivative. Equation (2) is the more common one in the nonlinear case but reduces to Eq. (1) in the linear case. This implies linear not only in having M but also R not a function of N.

The object of establishing these adjoint or associated equations is to enable a stationary expression to be constructed in which errors, etc., in N are acceptable. The purpose of the adjoint operator is to provide us with a commutation of fields that will allow us to write for any pair ψ^+ and N or more usually ψ^+ and δN that, when integrated over the domain of the system,

$$\iint \psi^+ A \psi \, dx \, dt = \iint \psi A^* \psi^+ \, dx \, dt \ .$$

Suitable boundary and continuity conditions are necessary in the case of differential operators such that the bilinear concomitant vanishes from this commutation.

The steps in establishing the adjoint equation then are two:

1. Replace the source with the appropriate adjoint source, mean or marginal as required, taken from the characteristic of interest.
2. Replace the forward operator with an adjoint operator, still if necessary a function of N and of either mean or marginal type as required.

And finally, as is well know, the general rules for creating the adjoint of an operator include:

1. Transpose a matrix.
2. Change the sign of odd-parity differential operators (and invoke suitably symmetric boundary and interface conditions when the forward conditions are of the "free" type).
3. Interchange the arguments of the kernels of integral operators.

How then shall this be described? My own suggestion for the latter part, the construction of the adjoint of a given operator to satisfy the commutation condition, is that we might speak of "reversing" the operator. My suggestion for naming the process of constructing either of the adjoint equations is that we might call this whole process the "adjoint construction"—mean or marginal as required.

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