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Letters to the Editor

Comments on "Stability Analysis of a Sampled-Data Controlled Nuclear Reactor System"

In the following remarks I hope to clarify some statements in a paper by Reisch[1] as well as add my own thoughts.

Figure 1 of Ref 1, showing the digital computer, is confusing in that a sampler is placed after a sampler. The "scanning device" appears to be the actual sampler in the system. It appears that the exit temperatures are collected from one complete scan, and the highest is selected to control the rod drives. This value is a single value of a continuous time function taken at the time the channel temperature was scanned. This interpretation is implied in Fig. 4. If this is the case, there should be no extra sampler shown since the computer would act as a zero-order hold and the transfer function of the computer could be written as

$$C(s) = \left[\frac{1 - \exp(-Ts)}{s}\right] \exp(-\tau s) \ , \tag{1}$$

where T is the scanning time, and τ is the delay time due to computer operations. If k_1/s is the transfer function of the control-rod drive, then the feedback transfer function of the system is

$$K(s) = \frac{k_1[1 - \exp(-Ts)]}{s^2} \exp(-\tau s) \ . \tag{2}$$

This transfer function is more representative of the system than that given in Eq. (1). If the computer-time delay is ignored, as is done in the article for simplicity, then z transforms can be used and this form of $K(s)$ does not add any complication to the system analysis. But if it is considered, one is almost forced to use modified z transforms to study the response of the system, although there are other ways of handling transport lag in a sampled-data system.

Under the general title "Some Basic Properties of z-Transform Theory," Eq. (12) is said to follow directly from Eq. (11), by using "the usual Laplace transform theory." If this were so then Eq. (12) should be

$$x^*(s) = \sum_{n=-\infty}^{\infty} x(nT) \exp(-nTs) \ . \tag{3}$$

The form of Eq. (12) results from the further assumption that $x(t) = 0$ for $t < 0$. Then from this form, the z transform is defined as

$$z = \exp(Ts) \tag{4}$$

in order to make $x(z)$ into an infinite series in z, a complex variable. Use of the equivalent form

$$s = (\ln z/T) \tag{5}$$

seems to imply that, if this quantity were substituted directly into the equation for $x^*(s)$, then $x(z)$ would directly follow. The function $x(z)$ must be written as an infinite series, and then it usually can be written in the closed form found in z-transform tables.

Use of the transformation given in Eq. (5) above explains the stability criterion stated as "for a stable sampled-data control system, the roots of the characteristic equation must be inside the unit circle in the complex z plane." The reason for this is that this transformation maps the left half of the s plane into the interior of the unit circle in the z plane. For an asympototically stable system all the real parts of the roots of the characteristic equation in the s plane must be negative. Therefore in the z plane they must be within the unit circle. This fact leads directly to analytical techniques for determining stability rather than the graphical procedure described under the section "Stability Analysis of the Sampled-Data Controlled Reactor System with the z-Transform Technique." As an example, by using the bilinear transformation[2]

$$z = \frac{1+w}{1-w} \ , \tag{6}$$

the unit circle is mapped into a strip in the left half of the w plane and yields an equation of the same order as the characteristic equation in the z variable. When this is done, Routh's criterion can be applied to ensure that no postive real roots exist.

Next, the methods of z-transform inversion are:

1) use of the inversion integral

$$x(nT) = (2\pi j)^{-1} \oint_{\Gamma} x(z) z^{n-1} \, dz \tag{7}$$

2) partial fraction expansion and recourse to the z-transform tables

3) power series expansion or long division.

Note should be taken of the fact that the inverse z transform of a function of z is not unique, so that inversion is a dangerous thing especially if z-transform tables are used as in method 2) above.

Also stated in the article is that z-transform methods are not applicable when the sampling frequency $1/T$ is less than twice the highest frequency occurring in the frequency characteristic of the continuous system. This criterion (Shannon's) is actually only applicable if the sampled signal is to be reconstructed accurately. Actually, the method is much more general than that, and stable sampled-data systems may be analyzed where Shannon's criterion does not hold. There is however, some danger of hidden oscillations since the inversion gives the signal value only at the

[1] F. REISCH, "Stability Analysis of a Sampled-Data Controlled Nuclear Reactor System," *Nucl. Sci. Eng.*, **26**, 378 (1966).

[2] J. T. LOU, *Digital and Sampled-Data Control Systems*, McGraw-Hill Book Co., Inc., New York, Toronto, London, pp. 244-247 (1959).

sampling instants[3]. These hidden oscillations can occur if the open-loop transfer function of the corresponding continuous system has complex poles. (The magnitude of the imaginary parts of the complex roots determines the sampling rate required to avoid hidden oscillations.) In this case the transfer function apparently does have such complex roots, and the criterion is necessary in order to see the oscillations in the output.

*Thomas J. Marciniak**

Argonne National Laboratory,
Argonne, Illinois

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*ANL-AMU Fellow, University of Notre Dame, Notre Dame, Indiana.

[3]E. I. JURY, "Hidden Oscillations in Sampled Data Control Systems," AIEE Trans., Pt. II, **76**, 391 (1956).

Reply to T. J. Marciniak's Comments on "Stability Analysis of a Sampled-Data Controlled Nuclear Reactor System"

The following remarks are in reply to the preceding Letter. Figures 1, 2, and 3 in the paper by Reisch[1] help in deriving Fig. 4 which is the usual form of a block diagram representing a sampled-data system. The sampler of Fig. 1 is identical with the sampler of Fig. 4. The comparator, which is connected to the scanning device and the memory unit, selects and places in the memory the highest of the two values: the temperature stored in the memory or the temperature of the cooling channel which is being scanned at that moment. In this way there is always only one value in the memory which, at the end of the checking cycle, equals the highest channel temperature during the cycle. This method does imply a variable time delay which can be handled by the modified z-transform method. Nevertheless, I preferred not to use it because my purpose was to demonstrate the usefulness of the application of modern control theories for reactor dynamic studies, and I think that the explanation and use of the modified z-transform technique would have extended the paper unnecessarily. Instead, I proposed the use of a scanning device sufficiently fast so that the variation of this time delay can be neglected.

[1]F. Reisch, *Nucl. Sci. Eng.*, **26**, 378 (1966).

To fully explain in one page "Some Basic Properties of the z-Transform Theory" is naturally impossible. I omitted both negative times and frequencies when deriving Eq. (12). It is stated properly that $x(z)$ is an infinite series, and it is left to the reader to understand that the z-transform tables mentioned contain these series in closed form.

My choice of the stability criterion in the z plane is deliberate. The roots of the characteristic equation can be calculated with sufficient accuracy by using double precision calculations which are available, according to my knowledge, in all types of modern computers and which are laborious to use. The coefficients of the characteristic equation are used even when the step responses of the sampled system are calculated in the time domain. There is no graphical procedure at all; both the roots of the characteristic equations and the time responses are calculated in the same run from the system parameters.

It is stated properly that the power series expansion method for inverse z transformation is sometimes called long division, i.e., there are two names for a single method. I agree that it would have been useful to point out that z-transform inversion is not unique and therefore must be handled with care.

The choice of the sampling frequency is a delicate feature. To be mathematically exact one should use a sampling frequency at least twice as high as the highest frequency of the continuous system. The fact is that the overwhelming parts of the existing systems have no frequency limit above which the system amplification disappears. Nevertheless the z-transform technique is widely used, for example, in the discussed paper. As high sampling frequencies are economically penalized, the problem is to find a formula for the lowest permissible sampling frequency without running the risk of hidden oscillations. My choice is somewhat arbitrary but I think it can be justified with a control technique background.

Frigyes Reisch

AB Atomenergi
Stockholm, Sweden

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