Letters to the Editor

On the Use of First-Order Perturbation Theory in Interface Shift Problems

In a recent Note,[1] Rahnema and Pomraning stated that the classic first-order perturbation formula, namely, Eq. (6) of Ref. 1, is incorrect for the internal interface perturbation problem, and they have proposed a "corrected" perturbation formula, namely, Eq. (17) of Ref. 1. They have recommended that the corrected perturbation formula be used for interface shift problems.

In this Letter we would like to make the following remarks on the above-mentioned conclusions.

1. An interface shift problem can be treated as either a material property perturbation (of order ϵ_M) or as a volume perturbation (of order ϵ_V). The classic first-order perturbation formula, Eq. (6) of Ref. 1, is correct to the first order in ϵ_M. The corrected perturbation formula, Eq. (17) of Ref. 1, is correct to the first order in ϵ_V. Both are different aspects of the same problem and there is nothing "incorrect" in either formula provided there is a clear understanding of what is implied by the term "first order."

2. In a perturbation problem in which the interface shift is great but the difference in material properties on either side of the interface is small, it is more accurate to use the classic first-order perturbation formula. On the other hand, in problems in which the interface shift is small but the change in material properties is great, it is more accurate to use the corrected perturbation formula. The relative accuracies of the two formulas depend on the relative magnitudes of the material perturbation ϵ_M and the volume perturbation ϵ_V. Whichever procedure is used, it is always possible to obtain greater accuracy by calculating the higher order perturbation terms. The point of view of Ref. 1 that the use of the classical perturbation formula is incorrect is not justified.

To justify our remarks, consider the particular two-region slab reactor interface shift problem considered in Ref. 1. To show clearly the material perturbations involved in this problem we rewrite the basic equations.

The unperturbed eigenvalue problem is

$$\left.\begin{aligned} -D_L \frac{d^2\phi_0}{dz^2} + \sigma_a\phi_0 = \Lambda_0\nu\sigma_f\phi_0 \quad & 0 \leqslant z \leqslant \tfrac{1}{2} \\ -D_R \frac{d^2\phi_0}{dz^2} = 0 \quad & \tfrac{1}{2} \leqslant z \leqslant 1 \end{aligned}\right\} . \tag{1}$$

The perturbed eigenvalue problem is

$$\left.\begin{aligned} -D_L \frac{d^2\phi}{dz^2} + \sigma_a\phi = \Lambda\nu\sigma_f\phi \quad & 0 \leqslant z \leqslant \tfrac{1}{2} \\ -(D_R + \epsilon_M\delta D)\frac{d^2\phi}{dz^2} + \epsilon_M\delta\sigma_a\phi = \Lambda\epsilon_M\delta\nu\sigma_f\phi \quad & \tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2} + \epsilon_V \\ -D_R \frac{d^2\phi}{dz^2} = 0 \quad & \tfrac{1}{2} + \epsilon_V \leqslant z \leqslant 1 \end{aligned}\right\} . \tag{2}$$

Here,

$$\epsilon_M\delta D = D_L - D_R \ ,$$

$$\epsilon_M\delta\sigma_a = \sigma_a \ ,$$

and

$$\epsilon_M\delta\nu\sigma_f = \nu\sigma_f \ , \tag{3}$$

where ϵ_M is the perturbation parameter associated with material property changes.

For this problem, Rahnema and Pomraning stated that the perturbed gradient $\nabla\phi$ does not differ by order ϵ_V from the unperturbed gradient $\nabla\phi_0$ to the right of the unperturbed interface, and that, on account of this, the derivation of the classic first-order perturbation formula is invalid. We point out that it is only required that $\nabla\phi$ and $\nabla\phi_0$ differ by order ϵ_M for the classical derivation to be valid. That this is indeed so can be easily seen by considering the physics of the problem. Referring to Fig. 1, we note that the material in the region $\frac{1}{2} \leqslant z \leqslant \frac{1}{2} + \epsilon_V$ can be gradually changed to approach the material in the region $0 \leqslant z \leqslant \frac{1}{2}$ by varying ϵ_M gradually, so that the unperturbed flux and gradient gradually change to the perturbed values, and the discontinuity in the gradient at $z = \frac{1}{2}$ gradually disappears. This gradual change can be symbolically represented as

$$\phi_0 \to \phi \qquad \text{in } 0 \leqslant z \leqslant 1 \ , \tag{4}$$

$$\nabla\phi_{0L} \to \nabla\phi_L \qquad \text{in } 0 \leqslant z \leqslant \tfrac{1}{2} \ , \tag{5}$$

and

$$\nabla\phi_{0R} \to \left\{\begin{aligned} \nabla\phi_L \quad & \text{in } \tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2} + \epsilon_V \\ \nabla\phi_R \quad & \text{in } \tfrac{1}{2} + \epsilon_V \leqslant z \leqslant 1 \end{aligned}\right\} . \tag{6}$$

Thus, it follows that for this material perturbation problem,

$$\phi = \phi_0 + O(\epsilon_M) \qquad \text{in } 0 \leqslant z \leqslant 1 \ , \tag{7}$$

$$\nabla\phi_L = \left\{\begin{aligned} \nabla\phi_{0L} + O(\epsilon_M) \quad & \text{in } 0 \leqslant z \leqslant \tfrac{1}{2} \\ \nabla\phi_{0R} + O(\epsilon_M) \quad & \text{in } \tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2} + \epsilon_V \end{aligned}\right\} , \tag{8}$$

[1] F. RAHNEMA and G. C. POMRANING, *Nucl. Sci. Eng.*, **78**, 393 (1981).

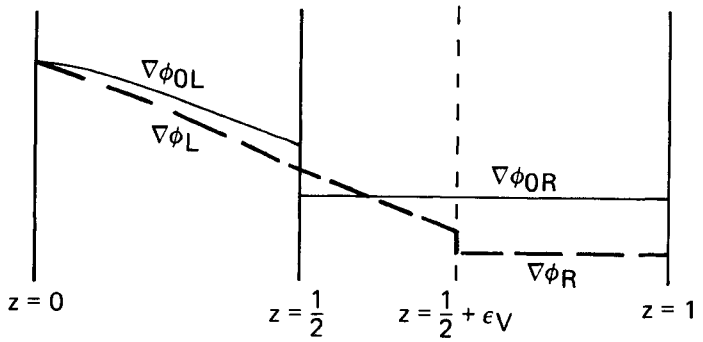

Fig. 1.

and

$$\nabla\phi_R = \nabla\phi_{0R} + O(\epsilon_M) \quad \text{in } \tfrac{1}{2} + \epsilon_V \leqslant z \leqslant 1 \ . \tag{9}$$

Hence, if the interface shift perturbation is treated as a material perturbation, then, as justified by our Eq. (8) and in fact demanded by the rigorous derivation of the classical first-order perturbation formula, $\nabla\phi$ should be replaced by $\nabla\phi_{0R}$ in the integral over the perturbed volume. If this is done, the exact Eq. (15) of Ref. 1 will reduce to the classic first-order perturbation formula, Eq. (6) of Ref. 1, which will give results correct to the first order in ϵ_M.

Parenthetically, we remark that the requirement of current continuity across the interface causes a discontinuity in $V\phi_0$ at $z = \frac{1}{2}$. However, we see that

$$\nabla\phi_{0R}\left(z = \tfrac{1}{2}\right) = \frac{D_L}{D_R}\nabla\phi_{0L}\left(z = \tfrac{1}{2}\right) = \left(1 + \epsilon_M \frac{\delta D}{D_R}\right)\nabla\phi_{0L}\left(z = \tfrac{1}{2}\right)$$
$$= \nabla\phi_{0L}\left(z = \tfrac{1}{2}\right) + O(\epsilon_M) \ . \tag{10}$$

Thus, this discontinuity is of order ϵ_M only, and this cannot cause any inconsistency in our Eq. (8).

For the corrected perturbation formula, we see from Fig. 1 that, by Taylor series expansion,

$$\nabla\phi_L = \nabla\phi_L\left(z = \tfrac{1}{2}\right) + O(\epsilon_V) = \nabla\phi_{0L}\left(z = \tfrac{1}{2}\right) + O(\epsilon_V)$$
$$\text{in } \tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2} + \epsilon_V \ . \tag{11}$$

Further, as also noted by Rahnema and Pomraning,

$$\nabla\phi_L \neq \nabla\phi_{0R} + O(\epsilon_V) \quad \text{in } \tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2} + \epsilon_V \ . \tag{12}$$

Thus, in the region of perturbation, $\nabla\phi$ ($\equiv \nabla\phi_L$) only approaches $\nabla\phi_{0L}(z = \frac{1}{2})$ and not $\nabla\phi_{0R}(z = \frac{1}{2})$ as $\epsilon_V \to 0$. If the interface shift perturbation is treated as a volume perturbation (of order ϵ_V), then the exact Eq. (15) of Ref. 1 should be converted to a surface integral over the unperturbed surface and $\nabla\phi$ replaced by $\nabla\phi_{0L}(z = \frac{1}{2})$ to get the corrected perturbation formula, Eq. (17) of Ref. 1, which will give results correct to the first order in ϵ_V.

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November 29, 1982

Reply to "On the Use of First-Order Perturbation Theory in Interface Shift Problems"

Rao and Lee[1] seem to have attributed more to our Note[2] than we intended. We did not, as these authors seem to imply, claim that the classic derivation of the standard first-order perturbation formula is incorrect. Indeed, the traditional derivation, equation, and interpretation of classic first-order perturbation theory are all entirely correct if the perturbed problem differs from the unperturbed one by order ϵ_M (presumed to be small). Our only purpose was to point out that this classic result is, in fact, *limited* to perturbations of order ϵ_M. More specifically, the classical first-order perturbation formula will not correctly treat the case of a slight internal interface shift, characterized as an order ϵ_V perturbation. This is so in spite of the fact that ϕ_0 and ϕ differ by a small amount, of order ϵ_V, for this class of problems.

It appears that all of the arguments and the analysis of Rao and Lee[1] do no more than repeat the arguments we have made,[2] in a somewhat different (and to us, more confusing) language. It might also be useful in this interchange to point out that a quite general perturbation formula, correct to first order in both ϵ_M and ϵ_V, has recently been obtained.[3] This new formula reduces to the classical first-order perturbation formula for perturbations of order ϵ_M to Eq. (17) of Ref. 2 for the order ϵ_V interface shift problem and, in general, correctly treats an arbitrary perturbation in first order, which alters the scalar flux and current by an order ϵ amount.

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January 1, 1983

[1] J. V. MURALIDHAR RAO and S. M. LEE, *Nucl. Sci. Eng.*, **84**, 72 (1983).
[2] F. RAHNEMA and G. C. POMRANING, *Nucl. Sci. Eng.*, **78**, 393 (1981).
[3] G. C. POMRANING, *Nucl. Sci. Eng.*, **83**, 72 (1983).