With (7-3) it follows that

$$\frac{1}{4}c^2 \int_{-1}^{+1} \nu \left[\int_{-1}^{+1} \nu'\phi(\nu,\nu') \left\{ \fint_{-1}^{+1} \frac{\mu d\mu}{(\nu-\mu)(\nu'-\mu)} \right\} d\nu'\right] d\nu$$

$$= -\frac{1}{2}c \int_{-1}^{+1} \nu \left[\fint_{-1}^{+1} \frac{\nu'\lambda(\nu')}{\nu-\nu'} \phi(\nu,\nu')d\nu'\right] d\nu$$

$$+\frac{1}{2}c \int_{-1}^{+1} \nu\lambda(\nu) \left[\fint_{-1}^{+1} \frac{\nu'}{\nu-\nu'} \phi(\nu,\nu')d\nu'\right] d\nu$$

$$+\frac{1}{4}\pi^2 c^2 \int_{-1}^{+1} \nu^3 \left[\int_{-1}^{+1} \phi(\nu,\nu')\delta(\nu'-\nu)d\nu'\right] d\nu \ . \tag{9}$$

In the second term in (6) one may interchange the order of the integrations with respect to μ and ν'. The integral over ν' can be evaluated. After the substitution of ν' for μ under the integral sign one finds:

$$\frac{1}{2}c \int_{-1}^{+1} \nu \left[\int_{-1}^{+1} \lambda(\nu')\phi(\nu,\nu') \left\{ \int_{-1}^{+1} \frac{\mu}{\nu-\mu} \times \delta(\nu'-\mu)d\mu \right\} d\nu'\right] d\nu$$

$$= \frac{1}{2}c \int_{-1}^{+1} \nu \left[\fint_{-1}^{+1} \frac{\nu'\lambda(\nu')}{\nu-\nu'} \phi(\nu,\nu')d\nu'\right] d\nu. \tag{10}$$

For the third term in (6) one finds in an analogous way:

$$\frac{1}{2}c \int_{-1}^{+1} \lambda(\nu) \left[\int_{-1}^{+1} \nu'\phi(\nu,\nu') \left\{ \fint_{-1}^{+1} \frac{\mu}{\nu'-\mu} \times \delta(\nu-\mu)d\mu \right\} d\nu'\right] d\nu$$

$$= \frac{1}{2}c \int_{-1}^{+1} \nu\lambda(\nu) \left[\fint_{-1}^{+1} \frac{\nu'}{\nu'-\nu} \phi(\nu,\nu')d\nu'\right] d\nu. \tag{11}$$

The fourth term in (6) gives according to definition 5:

$$\int_{-1}^{+1} \lambda(\nu) \left[\int_{-1}^{+1} \lambda(\nu')\phi(\nu,\nu') \left\{ \int_{-1}^{+1} \mu\delta(\nu-\mu)\times\delta(\nu'-\mu)d\mu \right\} d\nu'\right] d\nu$$

$$= \int_{-1}^{+1} \nu\lambda^2(\nu) \left[\int_{-1}^{+1} \delta(\nu'-\nu)\phi(\nu,\nu')d\nu'\right] d\nu. \tag{12}$$

With the results (9) - (12) one derives the following expression for the left-hand side of the relation (5):

$$\left\langle \int_{-1}^{+1} \mu(T(\nu)\times T(\nu'))_\mu \, d\mu, \ \phi(\nu,\nu') \right\rangle_2$$

$$= \int_{-1}^{+1} N(\nu) \left[\int_{-1}^{+1} \delta(\nu'-\nu)\phi(\nu,\nu')d\nu'\right] d\nu \tag{13}$$

with $N(\nu) = \nu[\lambda^2(\nu) + (\frac{1}{2}\pi c\nu)^2]$.

According to definition 2 the right-hand side of the identity (13) is equivalent to

$$\left\langle 1(\nu), \ \left\langle \delta(\nu'-\nu), \ N^{\frac{1}{2}}(\nu)\, N^{\frac{1}{2}}(\nu')\, \phi(\nu,\nu')\right\rangle_1 \right\rangle_1 \ ,$$

which, according to definition 5, is in turn equivalent to

$$\left\langle 1(\nu)\times\delta(\nu'-\nu), \ N^{\frac{1}{2}}(\nu)\, N^{\frac{1}{2}}(\nu')\, \phi(\nu,\nu')\right\rangle_2.$$

This proves the relation (5) for every $\phi\epsilon D_2$.

We remark that the relation (5) can also be proved starting from the fact that the generalized function $[T(\nu) \times T(\nu')]_\mu$ is a continuous function of the parameter μ in the sense of Ref. 4, Kap. 1 Anhang 2. This enables one to write the left-hand side of (5) as

$$\int_{-1}^{+1} \mu \left\langle (T(\nu)\times T(\nu'))_\mu, \ \phi(\nu,\nu')\right\rangle_2 d\mu \ ,$$

which is equivalent to

$$\int_{-1}^{+1} \mu \left\langle T_\mu(\nu), \ \left\langle T_\mu(\nu'), \ \phi(\nu,\nu')\right\rangle_1 \right\rangle_1 d\mu \ .$$

If one uses theorem 1 and the Poincaré-Bertrand formula (8) it is not difficult to show that this is equal to the expression in the right-hand side of (5) for every $\phi\epsilon D_2$.

Having established the results above one deduces all other formulae that have appeared in the literature, e.g. the full-range closure relation and the angular Green's function, using exactly analogous methods.

DISCUSSION

From the proof of theorem 2 above one concludes that McInerney's relation (Eq. (35)) is valid only if $\nu' \neq \nu$; in essence it corresponds to our relation (7-1). Therefore its use in the identity (Eq. (32)) is not justified, since (32) holds and is used also if $\nu' = \nu$. The correct form of (35) should contain a supplementary factor in the right-hand side to account for the contribution from the diagonal $\nu' = \nu$; this factor is essentially expressed in our result (7-2).

The statement of Kuščer and McCormick concerning the ambiguity in the Poincaré-Bertrand formula is due to a misinterpretation of the integrals occurring in this formula. In the notation of Ref. 3 the meaning of the integral over ν in the right-hand side of Eq. (2B) when $\mu' = \mu$ is uniquely defined by the factor $\pi^2 F(\mu,\mu)$ representing the contribution to the integral over μ' at the point $\mu' = \mu$. As we demonstrated above this contribution is equal to $\pi^2 F(\mu,\mu)\delta(\mu-\mu')$.

In conclusion we should like to make the following remarks.

The introduction of generalized functions in neutron transport theory requires proper definitions and their proper handling as functionals. Though the theorems may be stated in the usual shorthand notation, proofs should always be given with reference to the space of test-functions.

The symbol T for a generalized function is preferred to ϕ since in mathematical literature the symbol ϕ is commonly used to denote a test-function.

The symbol P only denotes a meaningful operator if placed before an integral sign. Therefore formulae like (2A) and (2B) in Ref. 3 are mathematically senseless.

The same warning is appropriate to formulae (4) and (7) in Ref. 2, where one silently passes from generalized functions ϵD_1^* in the left-hand side to generalized functions ϵD_2^* in the right-hand side. Such ambiguities cause confusion and should therefore be avoided.

To summarize we have given a rigorous proof that there is only one consistent system of formulae in neutron transport theory. It is the system that is currently used in this field, following the work of Case et al.

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Comment to the Preceding Letter by Kaper

We appreciate the effort of Kaper to mathematically justify what we hoped to convey in a heuristic manner. It is indeed reassuring to see that his Eq. (7-3) follows from our Eq. (3A), and that his derivation is closely related to ours (so that it seems to the same extent arbitrary).

We admit not having explained one symbol which has created doubt about the mathematical sense of the equations. Our $\int P$ stands for $P\int$ or $\fint$ of other authors, whereas

P alone (which of course is no operator) refers to future integration. Never did we make the statement, of which we are accused, that the Poincaré-Bertrand formula is ambiguous, but only that one of the integrals involved requires an interpretation, and that with a different interpretation the shape of the formula changes. Kaper too quotes the usual interpretation, equivalent to our Eq. (3B), after his Eq. (8).

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Further Comments on the Use of Generalized Functions in Neutron Transport Theory

We are indeed indebted to Mr. Kaper[1] for his clarification of the arguments introdueed by Jacobs and McInerney[2] and Kuščer and McCormick[3] regarding the angular Green's function of neutron transport theory. However, we believe it is unfortunate that there is a continued use of the obscure notation of the Poincaré-Bertrand formula[4] which we believe has led to all the difficulties.

In these further comments we shall add two points to the discussion: 1) We shall derive expression (7-3) of reference 1 without recourse to the Poincaré-Bertrand formula and in so-doing obtain an alternative (correct in the sense of distributions) to equation (3A) of reference 3. 2) For the sake of those who do not desire to become intimatelv acquainted with the involvements of distribution theory, we shall attempt to discuss the practical implication of these considerations.

Referring to reference 5 for mathematical background and notation we state two easily proven lemmas:

Lemma 1: $(x+i0)^{-1} = x^{-1} - i\pi\,\delta(x)$,

where x^{-1} is the canonical regularization (i.e. the Cauchy principal value).

Lemma 2: $\mu(\nu'-\mu+i0)^{-1}\;(\nu-\mu+i0)^{-1}$

$$= (\nu-\nu'+i0)^{-1}\;(\nu'(\nu'-\mu+i0)^{-1} - \nu(\nu-\mu+i0)^{-1}),$$

where all terms are distributions with respect to the three variables μ, ν' and ν. We use these lemmas to prove a theorem of central importance in the consideration of generalized functions relevant to transport theory.

Theorem: $\mu(\nu'-\mu)^{-1}(\nu-\mu)^{-1}$

$$= (\nu-\nu')^{-1}(\nu'(\nu'-\mu)^{-1} - \nu(\nu-\mu)^{-1}) + \pi^2\mu\delta(\nu'-\mu)\delta(\nu-\mu)$$

With change of variable notation, this result should be compared with equation (3A) of reference 3. Here, specifications on integration order are meaningless. A particular example of the use of this theorem is its application to the unit test function on (-1,+1) with respect to the variable μ. The result is

$$\int_{-1}^{+1} d\mu(\,\mu(\nu'-\mu)^{-1}(\nu-\mu)^{-1}) = (\nu-\nu')^{-1}\left(\nu'\ln\left[\frac{\nu'+1}{\nu'-1}\right] - \nu\ln\left[\frac{\nu+1}{\nu-1}\right]\right)$$
$$+ \pi^2\nu\delta(\nu-\nu')$$

when ν, ν' are in (-1,+1). This is in agreement with equation (7-3). However, it is obtained without the rather confusing arguments related between equations (8) and (7-3) of reference 1.

With regard to the practical implication of these considerations, we believe that it is important to state that crucial effects of clear notation (in the use of generalized functions in transport theory) occur only in problems where there is a source singular in the angle variable (i.e. the angular Green's function). Even in the case of such problems, a consistent use of the closure condition yields answers which can be correctly interpreted[2,3]. However, as Kaper points out, the notation of these answers may not be that of formal distribution theory (viz., that in reference 5).

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[1] H. G. KAPER, Letter to the Editors, *Nucl. Sci. Eng.* this issue, p. 423.

[2] A. M. JACOBS and J. J. McINERNEY, *Nucl. Sci. Eng.* 22, 119 (1965).

[3] I. KUŠČER and N. J. McCORMICK, *Nucl. Sci. Eng.* 23, 404 (1965).

[4] N. I. MUSKHELISHVILI, *Singular Integral Equations*, Noordhoff, Groningen (1953).

[5] I. M. GEL'FAND and G. E. SHILOV, *Generalized Functions*, Vol. I, Academic Press (1963).

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